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Locally Nash groups

Memoria presentada para optar al grado de
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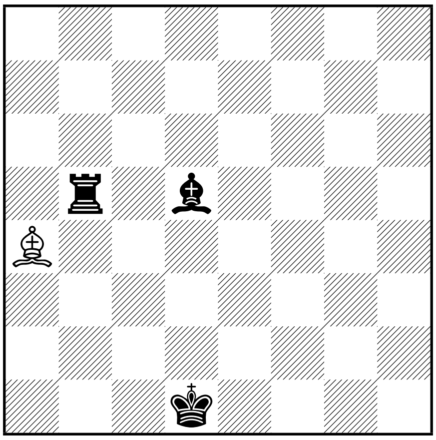

Raymond Smullyan, *Manchester Guardian, 1957.*
Recopilado en *The Chess Mysteries of the Arabian Knights.*

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Introducción

El objetivo de esta tesis es estudiar los grupos localmente Nash (GLN). El concepto de GLN se basa en el de función de Nash, que es el nombre que reciben las funciones que son a la vez analíticas reales y algebraicas. Las funciones de Nash han sido muy estudiadas en geometría algebraica real y analítica desde que J. Nash las estudiara en [**28**] en 1952 (véanse por ejemplo [**6**, §8], [**5**] y [**13**]). Un grupo de Nash es una variedad de Nash cuyas operaciones de grupo (multiplicación e inversión) son funciones de Nash. Los grupos de Nash comienzan a estudiarse en 1988, cuando A. Pillay demuestra en [**37**] que todo grupo definible sobre una estructura o-minimal sobre el cuerpo de los reales es un grupo localmente euclídeo y por lo tanto, por la solución de Gleason, Montgomery y Zippin del 5º problema de Hilbert, un grupo de Lie. En particular, un grupo semialgebraico (es decir, un grupo definible sobre la estructura de cuerpo de los reales) es un grupo de Nash. Desde 1990, los grupos de Nash han sido estudiados por diferentes autores usando a la vez técnicas de geometría algebraica real y analítica y de geometría o-minimal (véase por ejemplo M. Shiota [**44**]). Estos grupos, a pesar de ser objetos naturales en ambas áreas, sólo han sido clasificados en el caso unidimensional (por J.J. Madden y C.M. Stanton en [**26**] in 1992). Todo el trabajo realizado en grupos de Nash está basado en el estudio de los GLN. Esto ocurre ya que, al igual que como sucede con los grupos de Lie, el caso simplemente conexo es más fácil de manejar, sin embargo el recubridor universal de un grupo de Nash no es en general un grupo de Nash, sino un GLN. Por otra parte, por los resultados de E. Hrushovski y A. Pillay en [**19**], la clase de los GLN simplemente conexos es la clase de los recubridores universales de las componentes conexas (de la identidad) de los grupos algebraicos reales.

El resultado principal de esta tesis es la clasificación de los GLN abelianos de dimensión 2. Esta clasificación es en particular un primer paso para clasificar los grupos algebraicos reales abelianos de dimensión 2.

Todo GLN, abeliano, simplemente conexo y de dimensión n es analíticamente isomorfo a $(\mathbb{R}^n, +)$, por lo que su estructura de grupo localmente Nash queda determinada por una carta de Nash de la identidad. Madden y Stanton demostraron en [**26**] que las cartas de Nash de los GLN de dimensión 1 son exactamente las funciones id, exp, sen y $\wp$ de Weierstrass (véase el Aserto 5.6). La clasificación del caso unidimensional se basa en los siguientes tres resultados (los dos primeros debidos a K. Weierstrass).

Resultado de extensión: el germen de una función analítica que satisfaga un *teorema de adición algebraica* (TAA, véase la definición más adelante) puede ser transformado, mediante operaciones algebraicas, en el germen de una función analítica global que también satisface un TAA.

Resultado de descripción compleja: la descripción de las funciones de una variable y meromorfas en $\mathbb{C}$ que satisfacen un TAA.

Resultado de $\mathbb{C}$ versus $\mathbb{R}$: la relación entre el caso real y el complejo.

El núcleo de esta tesis consiste en resolver los problemas que aparecen al tratar de generalizar los tres resultados anteriores a dimensiones superiores. Antes de introducir el *resultado de extensión*, es necesario resaltar que es debido al *resultado de descripción compleja* por lo que hemos introducido el concepto análogo a GLN en la categoría compleja, al que denominamos grupo localmente $\mathbb{C}$-Nash (GL$\mathbb{C}$N). De esta manera, hablaremos de grupos localmente $\mathbb{K}$-Nash (GL$\mathbb{K}$N) para $\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$. El concepto de GL$\mathbb{C}$N está basado en el concepto de función $\mathbb{C}$-Nash, que ha sido estudiado por Y. Peterzil y S. Starchenko [**32, 33**], J. Adamus y S. Randriambololona [**3**], A. Tancredi y A. Tognoli [**46**], y otros autores.

El Capítulo 1 está dedicado a demostrar la versión n-dimensional del *resultado de extensión*. Denotamos por $\mathcal{O}_{\mathbb{K},n}$ el anillo de series de potencias en n variables con coeficientes en $\mathbb{K}$ ($= \mathbb{R}$ o $\mathbb{C}$) que son convergentes en algún entorno del origen, y por $\mathcal{M}_{\mathbb{K},n}$ su cuerpo de fracciones.

(TAA): *Sean u y v variables de $\mathbb{K}^n$. Diremos que $(\phi_1, \ldots, \phi_n) \in \mathcal{M}_{\mathbb{K},n}^n$ satisface un* teorema de adición algebraica *si $\phi_1, \ldots, \phi_n$ son algebraicamente independientes sobre $\mathbb{K}$ y si cada $\phi_i(u+v)$, $i = 1, \ldots, n$, es algebraico sobre*

$$\mathbb{K}(\phi_1(u), \ldots, \phi_n(u), \phi_1(v), \ldots, \phi_n(v)).$$

El concepto de TAA fue introducido por K. Weierstrass durante sus cursos sobre funciones abelianas en Berlín en 1870. Weierstrass enunció, sin demostración, que las funciones coordenadas de cualquier función meromorfa *global* que satisfaga un TAA deben ser o funciones abelianas o casos degenerados de éstas. Este enunciado de Weierstrass fue demostrado por F. Severi en [**42**] (véase también Y. Abe [**1, 2**]). (Más adelante en esta introducción daremos las versiones del enunciado de Weierstrass para dimensiones 1 y 2).

En su curso de Berlín, Weierstrass demostró el *resultado de extensión* (para una variable) y enunció, sin demostración, su correspondiente generalización para varias variables. Hasta donde sabemos, no hay publicada ninguna demostración de la generalización para varias variables. En esta tesis obtenemos dicho resultado como consecuencia (Corolario 1.12) del resultado principal del Capítulo 1, que enunciamos a continuación.

(La numeración de los resultados de esta introducción se corresponde con la posición en la que aparecen en esta memoria.)

Teorema 1.11 (Teorema de extensión). *Sea $\phi := (\phi_1, \ldots, \phi_n) \in \mathcal{M}_{\mathbb{K},n}^n$ una n-tupla que satisface un TAA. Entonces existen $\psi := (\psi_1, \ldots, \psi_n) \in$*

$\mathscr{M}^n_{\mathbb{K},n}$, *algebraica sobre* $\mathbb{K}(\phi)$ *y que también satisface un TAA, y una serie meromorfa adicional* $\psi_0 \in \mathscr{M}_{\mathbb{K},n}$*, algebraica sobre* $\mathbb{K}(\psi)$*, tales que:*

(1) *Para cada* $f(u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$,

  (*a*) $f(u+v) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u), \psi_0(v), \ldots, \psi_n(v))$ *y*

  (*b*) $f(-u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$.

(2) *Cada* ψ_i, $i = 0, \ldots, n$, *es el cociente de dos series de potencias, cada una de ellas convergente en todo* $\mathbb{C}^n$.

Corolario 1.12. *Toda* $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ *que satisfaga un TAA es algebraica sobre* $\mathbb{K}(\psi)$ *para una cierta* $\psi \in \mathscr{M}^n_{\mathbb{K},n}$ *que también satisface un TAA y cuyas funciones coordenadas son el cociente de dos series de potencias en* n *variables, cada una de ellas convergente en todo* $\mathbb{C}^n$.

En el Capítulo 2 desarrollamos la categoría de los GL$\mathbb{K}$N ($\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$) y en el Capítulo 3 estudiamos en más detalle el caso abeliano. El Teorema de extensión anteriormente mencionado nos permite caracterizar los GL$\mathbb{K}$N abelianos como sigue: Una función meromorfa $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^n$ se dice *invariante* (con respecto a la conjugación compleja) si $\overline{f(\overline{u})} = f(u)$. Diremos que una función $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^n$ es $\mathbb{C}$-*meromorfa* si es meromorfa y es $\mathbb{R}$-*meromorfa* si es meromorfa invariante, es decir, si se restringe a una función $f|_{\mathbb{R}^m} : \mathbb{R}^m \dashrightarrow \mathbb{R}^n$ meromorfa. Denotamos por $(\mathbb{K}^n, +, f)$ el GL$\mathbb{K}$N formado por $(\mathbb{K}^n, +)$ y una carta de la identidad que es restricción de una translación de la función $\mathbb{K}$-meromorfa f, es decir, es $f(u+a)$ para un cierto $a \in \mathbb{K}^n$. El siguiente resultado muestra que todo GL$\mathbb{K}$N abeliano y simplemente conexo es de esta forma (véase también el Aserto 3.7).

Teorema 3.8. *Todo grupo localmente* $\mathbb{K}$*-Nash abeliano, simplemente conexo y de dimensión* n *es isomorfo a* $(\mathbb{K}^n, +, f)$ *para alguna función* $\mathbb{K}$*-meromorfa* $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ *que satisface un TAA.*

Utilizando el Teorema 3.8 podemos caracterizar las clases de isomorfía de las estructuras localmente $\mathbb{K}$-Nash sobre el grupo analítico $(\mathbb{K}^n, +)$. De esta manera podemos dar una descripción de las clases de isomorfía de los GL$\mathbb{K}$N abelianos como sigue:

Lema 3.9. *Sean* $f, g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ *funciones* $\mathbb{K}$*-meromorfas que satisfacen un TAA. Los grupos localmente* $\mathbb{K}$*-Nash* $(\mathbb{K}^n, +, f)$ *y* $(\mathbb{K}^n, +, g)$ *son isomorfos si y sólo si existe* $\alpha \in \mathrm{GL}_n(\mathbb{K})$ *tal que* $g \circ \alpha$ *es algebraico sobre* $\mathbb{K}(f)$.

Proposición 3.10. (*I*) *Todo grupo localmente* $\mathbb{K}$*-Nash abeliano, conexo y de dimensión* n *es isomorfo a* $(\mathbb{K}^n, +, f)/\Gamma$ *donde* $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ *es una función* $\mathbb{K}$*-meromorfa que satisface un TAA y* Γ *es un subgrupo discreto de* $(\mathbb{K}^n, +)$.

(*II*) *Dados subgrupos discretos* Γ_1 *y* Γ_2 *de* $(\mathbb{K}^n, +)$*, los grupos localmente* $\mathbb{K}$*-Nash* $(\mathbb{K}^n, +, \phi)/\Gamma_1$ *y* $(\mathbb{K}^n, +, \psi)/\Gamma_2$ *son isomorfos si y sólo si existe un isomorfismo* $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$ *tal que* $\alpha(\Gamma_1) = \Gamma_2$.

Claramente, el enunciado (*II*) es equivalente a

(*a*) *existe un isomorfismo* $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$ *y* (*b*) *existe un automorfismo* $\gamma : (\mathbb{K}^n, +, \psi) \to (\mathbb{K}^n, +, \psi)$ *tal que* $\gamma(\Gamma_2) = \alpha(\Gamma_1)$.

Por lo tanto, el tipo de isomorfía de los GL𝕂N abelianos queda completamente determinado a partir del caso simplemente conexo mediante los grupos de automorfismos de estos últimos.

En el Capítulo 4 consideramos los GLℂN de dimensiones 1 and 2. Descripciones de las funciones meromorfas *globales* que satisfacen un TAA fueron obtenidas por Weierstrass en [**49**] para una variable y por P. Painlevé en [**30, 31**] para dos variables. Estos resultados implican fuertemente el enunciado de Weierstrass en una y dos variables. La descripción de Painlevé generaliza el *resultado de descripción compleja* para dimensión 2 y es la base que nos permite obtener una clasificación de los GLℂN abelianos en el Teorema 4.28. (A su vez hacemos uso de los GLℂN para dar una demostración de la clasificación de los GLN unidimensionales.)

En el Capítulo 5 tratamos la generalización del *resultado* ℂ *versus* ℝ. Recordemos que en el caso unidimensional la estructura de GLℂN asociada a la función exp induce dos estructuras de GLN no isomorfas, las asociadas a las funciones seno y exponencial real, respectivamente, y que se corresponden con los recubridores universales del círculo y del grupo multiplicativo de los reales, respectivamente. La situación en dimensión 2 es más complicada porque tenemos que considerar las extensiones de los recubridores universales de curvas elípticas por grupos lineales. Las cartas de los GLℂN que aparecen en la clasificación del caso abeliano, simplemente conexo y de dimensión 2 vienen expresadas en términos de funciones elípticas de Weierstrass. La clasificación del caso real requiere determinar cuáles de esas cartas son invariantes y estudiar sus períodos. Dado $\omega \in \mathbb{C} \setminus \mathbb{R}$, denotamos por $\wp_\omega$, ζ_ω, σ_ω y $\widetilde{\sigma}_{\omega,\xi}$ a las funciones $\wp_{\langle 1,\omega\rangle_{\mathbb{Z}}}$, $\zeta_{\langle 1,\omega\rangle_{\mathbb{Z}}}$, $\sigma_{\langle 1,\omega\rangle_{\mathbb{Z}}}$ y $\widetilde{\sigma}_{\langle 1,\omega\rangle_{\mathbb{Z}},\xi}$ respectivamente (véase la Notación 4.1, en la página 51).

A continuación enunciamos el resultado principal de la tesis:

Teorema 5.10. (*I*) *Todo grupo localmente Nash abeliano, simplemente conexo y de dimensión* 2 *es isomorfo a un grupo de uno y sólo uno de los siguientes tipos:*

(1) *A un producto directo de grupos localmente Nash unidimensionales, dados por cartas* id, exp, sen *o* $\wp_{ai}$ *para algún* $a \in \mathbb{R}^*$.
(2) $(\mathbb{R}^2, +, (\wp_{ai}(u), v - \zeta_{ai}(u)))$, *para algún* $a \in \mathbb{R}^*$.
(3) $(\mathbb{R}^2, +, (\wp_{ai}(u), e^v \widetilde{\sigma}_{ai,\xi}(u)))$, *para algún* $a \in \mathbb{R}^*$ *y* $\xi \in \mathbb{R} \setminus \mathbb{Q}$.
(4) $\left(\mathbb{R}^2, +, \left(\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv}\widetilde{\sigma}_{ai,-\xi i}(u))\right)\right)$, *para algunos* $a \in \mathbb{R}^*$ *y* $\xi \in \mathbb{R} \setminus a\mathbb{Q}$.
(5) *Al recubridor universal de la componente conexa del conjunto de puntos reales de una superficie abeliana simple definida sobre* ℝ.

(*II*) *Las clases de isomorfía en cada uno de los tipos mencionados es la siguiente:*

(*i*) *Dos grupos del tipo* 1 *son isomorfos si y sólo si sus grupos factores son isomorfos, donde* $(\mathbb{R}, +, \wp_{ai})$ *y* $(\mathbb{R}, +, \wp_{bi})$ *son isomorfos si y sólo si* $a/b \in \mathbb{Q}^*$.
(*ii*) *Dos grupos del tipo* 2*, definidos por* a *y* b *respectivamente, son isomorfos si y sólo si* $a/b \in \mathbb{Q}^*$.

(*iii*) *Dos grupos del tipo* 3*, definidos por* (a, ξ_1) *y* (b, ξ_2) *respectivamente, son isomorfos si y sólo si* $a/b \in \mathbb{Q}$ *y* $\xi_2 \in \mathbb{Q} + \xi_1\mathbb{Q}^*$*.*
(*iv*) *Dos grupos del tipo* 4*, definidos por* (a, ξ_1) *y* (b, ξ_2) *respectivamente, son isomorfos si y sólo si* $a/b \in \mathbb{Q}$ *y* $\xi_2 \in a\mathbb{Q} + \xi_1\mathbb{Q}^*$*.*
(*v*) *Dos grupos del tipo* 5 *son isomorfos si y sólo si existe una isogenia entre sus correspondientes superficies abelianas.*

Una clasificación más explícita de los grupos del tipo 5 requiere el estudio de la clasificación de las superficies abelianas definidas sobre $\mathbb{C}$ (véase C. Birkenhake y H. Lange [**4**, Chapter 10]) y del problema de existencia de variedades abelianas simples definidas sobre $\mathbb{R}$ que dejan de ser simples al considerarse sobre $\mathbb{C}$ (Véase J. Huisman [**22**, Example 48]). Ambos temas quedan más allá del alcance de esta tesis.

A continuación describimos los grupos de automorfismos de los GLN, abelianos, simplemente conexos y de dimensión 2. Como ya hemos observado, estos automorfismos nos permiten determinar el caso bidimensional abeliano vía la Proposición 3.10, mencionada anteriormente. A cada par de retículos Ω_1 y Ω_2 de $(\mathbb{C}^2, +)$ que compartan un subretículo común les asociamos los números $[\Omega_2 : \Omega_1] \in \mathbb{Q}^*$ y $\mathfrak{qc}(\Omega_2, \Omega_1) \in \mathbb{C}$ (véase la Definición 4.26). Dados $a, b \in \mathbb{C}$, denotamos $\operatorname{diag}(a, b)$ la matriz diagonal 2×2 cuyas entradas son a y b, por $\operatorname{Diag}(A, B)$ el conjunto $\{\operatorname{diag}(a, b) \mid a \in A, b \in B\}$ y por $\operatorname{Aut}(\mathbb{R}^2, +, f)$ el grupo de automorfismos localmente Nash de $(\mathbb{R}^2, +, f)$.

Proposición 5.11. *Sea* $(\mathbb{R}^2, +, f)$ *un grupo localmente Nash. Entonces,* $\operatorname{Aut}(\mathbb{R}^2, +, f)$ *es uno de los siguientes:*

(1) $\mathrm{GL}_2(\mathbb{R})$*, if* $f = \mathrm{id} \times \mathrm{id}$*.*

(2) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{R}^*)$*, si* $f = \mathrm{id} \times g$ *con* $g = \exp$, $\sin$ *or* $\wp_{ai}$*, para algún* $a \in \mathbb{R}^*$*.*

(3) $Gl_2(\mathbb{Q})$ *si* $f = g \times g$*, con* $g = \exp$ *o* $\sin$*.*

(4) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$*, si* $f = \exp \times \sin$*,* $\wp_{ai} \times \exp$ *or* $\wp_{ai} \times \sin$*, para algún* $a \in \mathbb{R}^*$*.*

(5.1) $\mathrm{GL}_2(\mathbb{Q})$*, si* $f = \wp_{ai} \times \wp_{bi}$*, para algunos* $a, b \in \mathbb{R}^*$ *tales que* $a/b \in \mathbb{Q}^*$*.*

(5.2) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$*, si* $f = \wp_{ai} \times \wp_{bi}$*, for some* $a, b \in \mathbb{R}^*$ *such that* $a/b \notin \mathbb{Q}^*$*.*

(6) $\left\{ q \begin{pmatrix} 1 & 0 \\ \mathfrak{qc}(\Omega, q\Omega) & [\Omega : q\Omega]q^{-2} \end{pmatrix} \,\middle|\, q \in \mathbb{Q}^* \right\}$*, donde* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *para algún* $a \in \mathbb{R}^*$*, si* $f = (\wp_{ai}(u), v - \zeta_{ai}(u))$*.*

(7) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi\mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \,\middle|\, q \in \mathbb{Q}^* \right\}$*, donde* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *para algún* $a \in \mathbb{R}^*$ *y* $\xi \in \mathbb{R} \setminus \mathbb{Q}$*, si* $f = (\wp_{ai}(u), e^v \widetilde{\sigma}_{ai,\xi}(u))$*.*

(8) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi\mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \,\middle|\, q \in \mathbb{Q}^* \right\}$*, donde* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *para algún* $a \in \mathbb{R}^*$ *y* $\xi \in \mathbb{R} \setminus a\mathbb{Q}$*, si* $f = (\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv}\widetilde{\sigma}_{ai,-\xi i}(u)))$*.*

Una introducción más detallada de los resultados de esta memoria de tesis puede encontrarse al inicio de cada capítulo.

Introduction

The main aim of this thesis is to study locally Nash groups (LNG). The concept of LNG is based on the corresponding one of Nash function, that is, a funtion that is both algebraic and real analytic. Nash functions have been a very active area of research in real algebraic and analytic geometry since J. Nash's paper [**28**] in 1952 (see e.g. [**6**, §8], [**5**] and [**13**]). A Nash group is a Nash manifold whose group operations (multiplication and inversion) are Nash maps. The study of Nash groups begin in 1988, when A. Pillay showed in [**37**] that every group definable in an o-minimal structure over the real field is locally euclidean and so, by Gleason, Montgomery and Zippin's solution of Hilbert's 5th problem, a Lie group. In particular, a semialgebraic group, *i.e.*, a group definable over just the field structure of the reals, is a Nash group. Since 1990, Nash groups have been studied by different authors using tools of both real algebraic and analytic, and o-minimal geometry (see e.g. M. Shiota [**44**]). These groups, despite of being natural objects in both areas, have only been classified in dimension 1 (by J.J. Madden and C.M. Stanton in [**26**] in 1992). All work done on Nash groups is based on the study of LNG. For, as in the case of Lie groups, the simply-connected ones are easier to handle, however the universal covering of a Nash group is not longer Nash but LNG. On the other hand, by the results of E. Hrushovski and A. Pillay in [**19**], the class of simply connected LNG is the class of universal coverings of connected components (of the identity) of real algebraic groups.

The main result of the thesis is a classification of two-dimensional abelian LNG. This classification in particular provides a first step towards a classification of two-dimensional abelian real algebraic groups.

Every simply-connected n-dimensional abelian LNG is analytically isomorphic to $(\mathbb{R}^n, +)$, so its Nash structure depends only on a Nash chart at 0. Madden and Stanton proved in [**26**] that the Nash charts of one-dimensional LNG are exactly id, exp, sin and a Weierstrass $\wp$-function (see Fact 5.6 below). This one-dimensional case depends upon the following three facts (the first two due to K. Weierstrass).

Extension fact: the germ of an analytic function admitting an *algebraic addition theorem* (AAT, see definition below) can be algebraically transformed into a germ of a global analytic function admitting an AAT.

Complex description fact: a description of one-variable meromorphic functions on $\mathbb{C}$ admitting an AAT.

$\mathbb{C}$ *vs.* $\mathbb{R}$ *fact:* relation between the complex and the real case.

The core of this thesis is solving the problems which arise trying to generalize these three facts to dimension greater than one. Before we speak of the *extension fact*, let us mention that it is because the *complex description fact* that we introduce the complex analogues of the LNG and we call them locally $\mathbb{C}$-Nash groups (L$\mathbb{C}$NG). So we will speak of locally $\mathbb{K}$-Nash groups (L$\mathbb{K}$NG) for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The concept of L$\mathbb{C}$NG is based on that of $\mathbb{C}$-Nash function, which in turn has been studied by Y. Peterzil and S. Starchenko [**32, 33**], J. Adamus and S. Randriambololona [**3**], A. Tancredi and A. Tognoli [**46**], and others.

Chapter 1 is dedicated to give an n-dimensional version of the *extension fact.* Let $\mathcal{O}_{\mathbb{K},n}$ be the ring of all power series in n variables with coefficients in $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) that are convergent on a neighborhood of the origin, and $\mathcal{M}_{\mathbb{K},n}$ its quotient field.

(AAT): *Let u and v be variables of $\mathbb{K}^n$. We say $(\phi_1, \ldots, \phi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admits an* algebraic addition theorem *if $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$ and if each $\phi_i(u+v)$, $i = 1, \ldots, n$, is algebraic over*

$$\mathbb{K}(\phi_1(u), \ldots, \phi_n(u), \phi_1(v), \ldots, \phi_n(v)).$$

The concept of AAT was introduced by K. Weierstrass during his lectures on abelian functions in Berlin in 1870. He stated, without a proof, that the coordinate functions of a *global* meromorphic map admitting an AAT are either abelian functions or degenerate abelian functions. Weierstrass statement was proved by F. Severi in [**42**] (see also Y. Abe [**1, 2**]). (We will go back to dimensions 1 and 2 of Weierstrass statement below in this introduction.)

Weierstrass proved the *extension fact* (one-dimensional) and stated, without a proof, an n-dimensional version of it. As far as we know, no such proof existed in the literature so far. We prove it here as a consequence (Corollary 1.12) of the main result of Chapter 1, that we now state.

(The numbering of the results in this introduction correspond to the place where they appear in the rest of this memoir.)

THEOREM 1.11 (Extension Theorem). *Let $\phi := (\phi_1, \ldots, \phi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ be a tuple admitting an AAT. Then, there exist $\psi := (\psi_1, \ldots, \psi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT and algebraic over $\mathbb{K}(\phi)$, and an additional meromorphic series $\psi_0 \in \mathcal{M}_{\mathbb{K},n}$ algebraic over $\mathbb{K}(\psi)$ such that:*

(1) *For each $f(u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$,*
  (a) *$f(u+v) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u), \psi_0(v), \ldots, \psi_n(v))$ and*
  (b) *$f(-u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$.*

(2) *Each ψ_i, $i = 0, \ldots, n$, is the quotient of two convergent power series whose complex domain of convergence is $\mathbb{C}^n$.*

COROLLARY 1.12. *Any $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT is algebraic over $\mathbb{K}(\psi)$, for some meromorphic series $\psi \in \mathcal{M}^n_{\mathbb{K},n}$ that admits an AAT and* *whose coordinate functions are the quotient of two convergent power series in n variables whose complex domain of convergence is $\mathbb{C}^n$.*

In Chapter 2, we develop the category of L$\mathbb{K}$NG ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and, in Chapter 3, we focus on abelian L$\mathbb{K}$NG. The Extension Theorem allow us to prove that abelian L$\mathbb{K}$NG can be characterized in terms of meromorphic maps in the following sense. A meromorphic map $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^n$ is an *invariant meromorphic map* (with respect to complex conjugation) if $\overline{f(\overline{u})} = f(u)$. We will say that a map $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^n$ is $\mathbb{C}$-*meromorphic* if it is just meromorphic and $\mathbb{R}$-*meromorphic* if it is invariant meromorphic, that is, it restricts to a meromorphic map $f|_{\mathbb{R}^m} : \mathbb{R}^m \dashrightarrow \mathbb{R}^n$. We will denote by $(\mathbb{K}^n, +, f)$ the L$\mathbb{K}$NG $(\mathbb{K}^n, +)$ with a chart at 0 given by a restriction of a translate of a $\mathbb{K}$-meromorphic map f, that is, by $f(u + a)$ for some $a \in \mathbb{K}^n$. The next result shows that every simply connected abelian L$\mathbb{K}$NG is of this form (see also Remark 3.7).

THEOREM 3.8. *Every simply connected n-dimensional abelian locally $\mathbb{K}$-Nash group is isomorphic to some $(\mathbb{K}^n, +, f)$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a $\mathbb{K}$-meromorphic map admitting an AAT.*

Making use of Theorem 3.8, we can caracterize the isomorphism type of a locally $\mathbb{K}$-Nash structure over the analytic group $(\mathbb{K}^n, +)$. This allows us to give a description of the isomorphims types of abelian L$\mathbb{K}$NG, as follows.

LEMMA 3.9. *Let $f, g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be $\mathbb{K}$-meromorphic maps admitting an AAT. Then, the locally $\mathbb{K}$-Nash groups $(\mathbb{K}^n, +, f)$ and $(\mathbb{K}^n, +, g)$ are isomorphic if and only if there exists $\alpha \in \mathrm{GL}_n(\mathbb{K})$ such that $g \circ \alpha$ is algebraic over $\mathbb{K}(f)$.*

PROPOSITION 3.10. (I) *Every connected n-dimensional abelian locally $\mathbb{K}$-Nash group is isomorphic to some $(\mathbb{K}^n, +, f)/\Gamma$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a $\mathbb{K}$-meromorphic map admitting an AAT and Γ is a discrete subgroup of $(\mathbb{K}^n, +)$.*

(II) *Given discrete subgroups Γ_1 and Γ_2 of $(\mathbb{K}^n, +)$, the locally $\mathbb{K}$-Nash groups $(\mathbb{K}^n, +, \phi)/\Gamma_1$ and $(\mathbb{K}^n, +, \psi)/\Gamma_2$ are isomorphic if and only if there exists an isomorphism $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$ such that $\alpha(\Gamma_1) = \Gamma_2$.*

Clearly, assertion (II) is equivalent to

- (a) *there exists a isomorphism $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$, and*
- (b) *there exists an automorphism $\gamma : (\mathbb{K}^n, +, \psi) \to (\mathbb{K}^n, +, \psi)$ such that $\gamma(\Gamma_2) = \alpha(\Gamma_1)$.*

Therefore, the study of abelian L$\mathbb{K}$NG relies on the simply connected ones and the automorphism groups of the latter.

In Chapter 4, we consider L$\mathbb{C}$NG of dimensions 1 and 2. Descriptions of *global* meromorphic maps admitting an AAT were given by Weierstrass in [**49**] in one-variable and P. Painlevé in [**30, 31**] in the two-variable case. These descriptions strongly imply Weierstrass mentioned statement for one and two-variables. The mentioned Painlevé's description takes care of the *complex description fact* in dimension 2 and it is the basic fact that allows us to obtain a classification of two-dimensional abelian L$\mathbb{C}$NG in Theorem 4.28.

(We also make use of the L$\mathbb{C}$NG to give an alternative proof of the classification of one-dimensional LNG.)

In Chapter 5, we will deal with the $\mathbb{C}$ *vs.* $\mathbb{R}$ *fact*. We recall that in the one-dimensional case the L$\mathbb{C}$NG associated to exp has two (non-isomorphic) real versions, associated to the function sin and (real) exp respectively (they correspond to the universal coverings of the circle group and the multiplicative group of the reals). The situation in dimension two is more complicated because we have to consider extensions of universal coverings of elliptic curves by linear groups. The charts of the L$\mathbb{C}$NG which appear in the classification of the simply connected abelian two-dimensional case are given in terms of Weierstrass elliptic functions. The classification of the real case is based on determining which of those charts are invariant and the study of their period groups. Given $\omega \in \mathbb{C} \setminus \mathbb{R}$, we denote by $\wp_\omega$, ζ_ω, σ_ω and $\widetilde{\sigma}_{\omega,\xi}$ the functions $\wp_{\langle 1,\omega\rangle_\mathbb{Z}}$, $\zeta_{\langle 1,\omega\rangle_\mathbb{Z}}$, $\sigma_{\langle 1,\omega\rangle_\mathbb{Z}}$ and $\widetilde{\sigma}_{\langle 1,\omega\rangle_\mathbb{Z},\xi}$, respectively (see Notation 4.1, page 51).

We now state the main result of this thesis.

Theorem 5.10. (*I*) *Every two-dimensional simply connected abelian locally Nash group is isomorphic to a group of one and only one of the following types:*

(1) *A direct product of one-dimensional locally Nash groups with charts* id, exp, sin *or* $\wp_{ai}$ *for some* $a \in \mathbb{R}^*$.
(2) $(\mathbb{R}^2, +, (\wp_{ai}(u), v - \zeta_{ai}(u)))$, *for some* $a \in \mathbb{R}^*$.
(3) $(\mathbb{R}^2, +, (\wp_{ai}(u), e^v \widetilde{\sigma}_{ai,\xi}(u)))$, *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus \mathbb{Q}$.
(4) $(\mathbb{R}^2, +, (\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv}\widetilde{\sigma}_{ai,-\xi i}(u))))$, *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus a\mathbb{Q}$.
(5) *The universal covering group of the connected component of the real points of a simple abelian surface defined over* $\mathbb{R}$.

(*II*) *The isomorphism classes within each type are defined as follows:*

(*i*) *Two groups of type* 1 *are isomorphic if and only if their factor groups are isomorphic, where* $(\mathbb{R}, +, \wp_{ai})$ *and* $(\mathbb{R}, +, \wp_{bi})$ *are isomorphic if and only if* $a/b \in \mathbb{Q}^*$.
(*ii*) *Two groups of type* 2, *defined by* a *and* b *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}^*$.
(*iii*) *Two groups of type* 3, *defined by* (a, ξ_1) *and* (b, ξ_2) *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}$ *and* $\xi_2 \in \mathbb{Q} + \xi_1\mathbb{Q}^*$.
(*iv*) *Two groups of type* 4, *defined by* (a, ξ_1) *and* (b, ξ_2) *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}$ *and* $\xi_2 \in a\mathbb{Q} + \xi_1\mathbb{Q}^*$.
(*v*) *Two groups of type* 5 *are isomorphic if and only if there is an isogeny between the corresponding abelian surfaces.*

A more explicit classification of groups of type 5 requires the study of both the classification of abelian surfaces defined over $\mathbb{C}$ (see C. Birkenhake and H. Lange [**4**, Chapter 10]) and the problem of having a simple abelian surface over $\mathbb{R}$ such that it is nonsimple when considered over $\mathbb{C}$ (see J. Huisman [**22**, Example 48]). Both topics go beyond the scope of this thesis.

We now describe the automorphism group of two-dimensional simply connected abelian LNG which, as we have mentioned above, will give us a description of two-dimensional abelian LNG via Proposition 3.10. Given a pair of lattices Ω_1 and Ω_2 of $(\mathbb{C}^2, +)$ with a common sublattice, we will associate two numbers $[\Omega_2 : \Omega_1] \in \mathbb{Q}^*$ and $\mathfrak{qc}(\Omega_2, \Omega_1) \in \mathbb{C}$ (see Definition 4.26). Given $a, b \in \mathbb{C}$, we denote $\operatorname{diag}(a, b)$ the diagonal 2×2 matrix whose entries in the diagonal are a and b, and $\operatorname{Diag}(A, B) := \{\operatorname{diag}(a, b) \mid a \in A, b \in B\}$. $\operatorname{Aut}(\mathbb{R}^2, +, f)$ denotes the group of locally Nash automorphisms of $(\mathbb{R}^2, +, f)$.

Proposition 5.11. *Let $(\mathbb{R}^2, +, f)$ be a locally Nash group. Then, $\operatorname{Aut}(\mathbb{R}^2, +, f)$ is one of the following:*

(1) $\mathrm{GL}_2(\mathbb{R})$, *if* $f = \mathrm{id} \times \mathrm{id}$.

(2) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{R}^*)$, *if* $f = \mathrm{id} \times g$ *with* $g = \exp, \sin$ *or* $\wp_{ai}$, *for some* $a \in \mathbb{R}^*$.

(3) $Gl_2(\mathbb{Q})$ *if* $f = g \times g$, *with* $g = \exp$ *or* $\sin$.

(4) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$, *if* $f = \exp \times \sin$, $\wp_{ai} \times \exp$ *or* $\wp_{ai} \times \sin$, *for some* $a \in \mathbb{R}^*$.

(5.1) $\mathrm{GL}_2(\mathbb{Q})$, *if* $f = \wp_{ai} \times \wp_{bi}$, *for some* $a, b \in \mathbb{R}^*$ *such that* $a/b \in \mathbb{Q}^*$.

(5.2) $\operatorname{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$, *if* $f = \wp_{ai} \times \wp_{bi}$, *for some* $a, b \in \mathbb{R}^*$ *such that* $a/b \notin \mathbb{Q}^*$.

(6) $\left\{ q \begin{pmatrix} 1 & 0 \\ \mathfrak{qc}(\Omega, q\Omega) & [\Omega : q\Omega] q^{-2} \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$, *if* $f = (\wp_{ai}(u), v - \zeta_{ai}(u))$.

(7) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi \mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus \mathbb{Q}$, *if* $f = (\wp_{ai}(u), e^v \widetilde{\sigma}_{ai,\xi}(u))$.

(8) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi \mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus a\mathbb{Q}$, *if* $f = (\wp_{ai}(u), \frac{1}{2i}(e^{iv} \widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv} \widetilde{\sigma}_{ai,-\xi i}(u)))$.

For a more detailed introduction to the results of this memoir, see the beginning of each chapter.

Chapter 1

Algebraic Addition Theorems

In this chapter we prove the Extension Theorem (Theorem 1.11), which will be essential to study abelian locally $\mathbb{K}$-Nash groups in Chapter 3. We have generalized some arguments given in the classical book of Hancock [**16**], where a proof of the one-dimensional case of Corollary 1.12 is given. We point out that Theorem 1.11 gives not only an extension result, but also a uniform rational version of the AAT. This will also allow us to compare abelian locally $\mathbb{C}$-Nash groups with complex algebraic groups in Chapter 3 (Theorem 3.12).

One of the key ideas of the mentioned proof in [**16**, Art.388] is that if $\varphi \in \mathscr{M}_{\mathbb{C},1}$ admits an AAT and $P(\varphi(u), \varphi(v), \varphi(u+v)) = 0$, for some $P \in \mathbb{C}[X, Y, Z]$, then each coefficient of $P(\varphi(u), \varphi(v), Z)$, as a polynomial in Z, gives rise, after a convenient evaluation, to a meromorphic function from $\mathbb{C}$ to $\mathbb{C}$ admitting an AAT and algebraic over $\mathbb{C}(\varphi)$. In Lemmas 1.8 and 1.10, we generalize this sort of arguments – concerning the coefficients – to the case of several variables. Then, we combine the results obtained with some ingredients from the theory of elliptic functions in order to obtain (2) in Theorem 1.11.

The chapter is divided as follows. In Section 1, we recall some basic properties of quotients of power series and their relation with germs of meromorphic maps. In Section 2, we include a proof of Fact 1.2, a generalization of the classical result about separation of variables: a complex analytic function $f(u, v)$ that is rational in u, for each fixed v, and rational in v, for each fixed u, belongs to $\mathbb{C}(u, v)$. Our version of this result, for germs of meromorphic maps of several variables, also takes into account the presence of a map algebraic over the others. Because of the latter, we will adapt the classical proof (in S. Bochner and W.T. Martin [**8**]) via specific techniques which appear in C.L. Siegel [**45**] (see Lemma 1.6 and the proof of Fact 1.2). In Section 3, we give the proof of Theorem 1.11, and we finish with some properties of differentials of maps admitting an AAT that will be useful in Chapter 3.

1. Preliminaries

We recall that $\mathcal{O}_{\mathbb{K},n}$ denotes the ring of all power series in n variables with coefficients in $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) that are convergent on a neighborhood of the origin and $\mathcal{M}_{\mathbb{K},n}$ its quotient field. For each $\epsilon > 0$ we define the open ball

$$U_{\mathbb{K},n}(\epsilon) := \{a \in \mathbb{K}^n \mid \|a\| < \epsilon\},$$

where $\| \; \|$ denotes the Euclidean norm (identifying $\mathbb{C}$ with $\mathbb{R}^2$). Note that through this memory, for technical reasons, we are using balls instead of polydiscs. We will only consider convergence over open subsets of $\mathbb{C}^n$ and we denote by $U_n(\epsilon)$ the open ball $U_{\mathbb{C},n}(\epsilon)$. We say that $(\phi_1, \ldots, \phi_m) \in \mathcal{M}_{\mathbb{K},n}^m$ is *convergent* in $U_n(\epsilon)$ if each $\phi_1, \ldots, \phi_m$ is the quotient of two power series convergent on $U_n(\epsilon)$.

Let us recall the relation between power series and (real) analytic functions. We will use R.C. Gunning and H. Rossi [**15**] and J.M. Ruiz [**39**] as basic references. Let $U \subset \mathbb{K}^n$ be an open connected neighborhood of 0. We denote by $\mathcal{O}_U$ the ring of all analytic functions on U and by $\mathcal{O}_{\mathbb{K}^n,0}$ the ring of germs of analytic functions at 0. The map

$$^{\mathrm{a}} : \mathcal{O}_{\mathbb{K},n} \to \mathcal{O}_{\mathbb{K}^n,0} : \phi \mapsto {}^{\mathrm{a}}\phi$$

mapping each ϕ to the germ of the analytic function

$$^{\mathrm{a}}\phi : U_\phi \to \mathbb{K} : a \mapsto \phi(a),$$

where $U_\phi \subset \mathbb{K}^n$ is an open neighborhood of 0 where ϕ converges, is an isomorphism of rings. For each f in $\mathcal{O}_U$ or in $\mathcal{O}_{\mathbb{K}^n,0}$ we denote by ${}^{\mathrm{t}}f$ its Taylor power series expansion at 0. The maps $^{\mathrm{a}}$ and $^{\mathrm{t}}$ are inverse maps of each other.

On the other hand, by the identity principle for analytic functions, both $\mathcal{O}_U$ and $\mathcal{O}_{\mathbb{K}^n,0}$ are integral domains. We denote by $\mathcal{M}_U$ and $\mathcal{M}_{\mathbb{K}^n,0}$ their respective quotient fields. The maps $^{\mathrm{a}}$ and $^{\mathrm{t}}$ are naturally defined for these quotients fields and give us also an isomorphism of $\mathcal{M}_{\mathbb{K},n}$ and $\mathcal{M}_{\mathbb{K}^n,0}$. Similarly, we define $^{\mathrm{a}}$ and $^{\mathrm{t}}$ for tuples.

Remark 1.1. A meromorphic function on an open subset $U \subset \mathbb{C}^n$ is a global section of the sheaf over U whose stalks at $x \in U$ are the quotient field of the germs of analytic functions at x. In other words, a meromorphic function on U is given by an open covering $\{U_i\}_{i \in I}$ of U and a collection of analytic functions $h_i, g_i : U_i \to \mathbb{C}$ such that

$$g_i \cdot h_j = g_j \cdot h_i \qquad \text{in } U_i \cap U_j \text{ for each } i, j \in I.$$

Although clearly the elements of $\mathcal{M}_U$ are meromorphic functions, the converse is not necessarily true. The problem of determining whether or not the converse holds for a certain U is known as the Poincaré problem. For example, it holds if $U = \mathbb{C}^n$ (see [**15**, Ch. VIII, §B, Corollary 10]).

We give a detailed formulation of the definition of algebraic addition theorem given in the introduction. Let $\epsilon > 0$. Let $\phi := (\phi_1, \ldots, \phi_m) \in \mathcal{M}_{\mathbb{K},n}^m$ be convergent on $U_n(\epsilon)$, let $a \in U_{\mathbb{K},n}(\epsilon)$ and let $(u, v) := (u_1, \ldots, u_n, v_1, \ldots, v_n)$ be a $2n$-tuple of variables. We will use the following notation:

$\phi_{(u,v)} := (\phi_1(u), \dots, \phi_m(u), \phi_1(v), \dots, \phi_m(v)) \in \mathscr{M}^{2m}_{\mathbb{K},2n}$.
$\phi_{u+v} := (\phi_1(u+v), \dots, \phi_m(u+v)) \in \mathscr{M}^{m}_{\mathbb{K},2n}$.
$\phi_{u+a} := (\phi_1(u+a), \dots, \phi_m(u+a)) \in \mathscr{M}^{m}_{\mathbb{K},n}$.

Given $\phi \in \mathscr{M}^n_{\mathbb{K},p}$ and $\psi \in \mathscr{M}^m_{\mathbb{K},p}$ we say that the tuple ϕ is *algebraic* over $\mathbb{K}(\psi) := \mathbb{K}(\psi_1, \dots, \psi_m)$ if each component, $\phi_1, \dots, \phi_n$, is algebraic over $\mathbb{K}(\psi)$.

Thus, $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ admits an algebraic addition theorem (AAT) *if $\phi_1, \dots, \phi_n$ are algebraically independent over $\mathbb{K}$ and ϕ_{u+v} is algebraic over $\mathbb{K}(\phi_{(u,v)})$.*

Note that if $\phi \in \mathscr{M}_{\mathbb{R},n}$ admits an AAT then ϕ also admits an AAT when considered as an element of $\mathscr{M}_{\mathbb{C},n}$.

2. Bochner's rationality result

This section is dedicated to give an algebraic proof of a particular case of a result of S. Bochner in [**7**, Theorem 3]. We thank M. Villarino for letting us know the existence of this result. The original statement of Bochner does not include the hypothesis of f being algebraic over $\mathbb{K}(\Psi_{(u,v)})$ (see below). However, in our context of functions admitting an AAT, it is a natural requirement. On the other hand, our proof gives an effective computation of the degrees of the relevant polynomials. Both proofs differ at the end, where we use some arguments from the theory of abelian functions and Bochner uses Baire's Category Theorem.

Fact 1.2. *Let $\epsilon > 0$. Let $\Psi := (\psi_0, \dots, \psi_n) \in \mathscr{M}^{n+1}_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ such that $\psi_1, \dots, \psi_n$ are algebraically independent over $\mathbb{K}$ and ψ_0 is algebraic over $\mathbb{K}(\psi_1, \dots, \psi_n)$. Let $f(u,v) \in \mathscr{M}_{\mathbb{K},2n}$ be convergent on $U_{2n}(\epsilon)$ and algebraic over $\mathbb{K}(\Psi_{(u,v)})$. If, for each $a \in U_{\mathbb{K},n}(\epsilon)$, both $f(u,a) \in \mathbb{K}(\Psi(u))$ and $f(a,v) \in \mathbb{K}(\Psi(v))$, then $f(u,v) \in \mathbb{K}(\Psi_{(u,v)})$.*

Before proving the fact we need some technical lemmas. Let $\epsilon > 0$. Let $u = (u_1, \dots, u_p)$ and $v = (u_1, \dots, u_q)$ be tuples of variables and let $\phi \in \mathscr{M}_{\mathbb{K},p+q}$ be convergent in $U_{p+q}(\epsilon)$, that is, $\phi(u,v) = \frac{\alpha(u,v)}{\beta(u,v)}$ for some $\alpha, \beta \in \mathscr{O}_{\mathbb{K},p+q}$ convergent on $U_{p+q}(\epsilon)$. Given a point $a \in \mathbb{K}^q$ we write $\phi(u,a) \in \mathscr{M}_{\mathbb{K},p}$ if $\beta(u,a) \neq 0$. The following is well known.

Lemma 1.3. *Let $\epsilon > 0$ and let $\phi := (\phi_1, \dots, \phi_m) \in \mathscr{M}^m_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$. Let $p, q \in \mathbb{N}$ be such that $p + q = n$. Then:*

(1) *There exists an open dense subset U of $U_{\mathbb{K},n}(\epsilon)$ such that*
$$^{\mathrm{a}}\phi : U \to \mathbb{K}^m : a \mapsto \phi(a)$$
is an analytic function.

(2) *There exists an open dense subset V of $U_{\mathbb{K},q}(\epsilon)$ such that*
$$V \subset \{a \in U_{\mathbb{K},q}(\epsilon) \mid \phi(u,a) \in \mathscr{M}^m_{\mathbb{K},p}\}.$$

(3) *If there exists an open subset W of $U_{\mathbb{K},q}(\epsilon)$ such that*
$$W \subset \{a \in U_{\mathbb{K},q}(\epsilon) \mid \phi(u,a) \in \mathscr{M}^m_{\mathbb{K},p} \text{ and } \phi(u,a) = 0\}$$
then $\phi = 0$.

Proof. For each $i \in \{1,\dots,m\}$, let $\alpha_i, \alpha_{m+i} \in \mathcal{O}_{\mathbb{K},n}$, $\alpha_{m+i} \neq 0$, be such that $\phi_i = \frac{\alpha_i}{\alpha_{m+i}}$. For each $i \in \{1,\dots,2m\}$, let ${}^{\mathrm{a}}\alpha_i : U_{\mathbb{K},n}(\epsilon) \to \mathbb{K} : a \mapsto \alpha_i(a)$. For (1), we note that since ${}^{\mathrm{a}}\alpha_{m+1},\dots,{}^{\mathrm{a}}\alpha_{2m}$ are analytic on $U_n(\epsilon)$ and not identically zero, the set

$$U := \{a \in U_{\mathbb{K},n}(\epsilon) \mid \alpha_{m+1}(a)\cdot \ldots \cdot \alpha_{2m}(a) \neq 0\}$$

is an open dense subset of $U_{\mathbb{K},n}(\epsilon)$ by the identity principle. For (2), it is enough to project the open set U. For (3), we note that since ${}^{\mathrm{a}}\alpha_1,\dots,{}^{\mathrm{a}}\alpha_m$ are analytic on U and identically zero on $\{(b,a) \in U \mid b \in W\}$, we get that $\alpha_1,\dots,\alpha_m = 0$. ☐

We next show that evaluation preserves algebraic relations in the following uniform sense.

Lemma 1.4. *Let $\epsilon > 0$. Let $\phi \in \mathcal{M}^m_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ and $\phi_1,\dots,\phi_m$ algebraically independent over $\mathbb{K}$. Let $f(u,v) \in \mathcal{M}_{\mathbb{K},2n}$ be convergent on $U_{2n}(\epsilon)$ and $f(u,a) \in \mathcal{M}_{\mathbb{K},n}$, for all $a \in U_{\mathbb{K},n}(\epsilon)$. If $f(u,v)$ is algebraic over $\mathbb{K}(\phi_{(u,v)})$ then $f(u,a)$ is algebraic over $\mathbb{K}(\phi(u))$, for each $a \in U_{\mathbb{K},n}(\epsilon)$. Furthermore, there exist $N \in \mathbb{N}$ and $\ell \leq N$ such that for each $a \in U_{\mathbb{K},n}(\epsilon)$, the minimal polynomial of $f(u,a)$ over $\mathbb{K}(\phi(u))$ can be written in the form*

$$Y^\ell + \sum_{i=0}^{\ell-1} \frac{R_i(\phi)}{S_i(\phi)} Y^i \quad R_i, S_i \in \mathbb{K}[X_1,\dots,X_m]^{\leq N}, S_i \neq 0,$$

where $\mathbb{K}[X_1,\dots,X_m]^{\leq N}$ denotes the polynomials of $\mathbb{K}[X_1,\dots,X_m]$ whose degree in each of the variables $X_1,\dots,X_m$ is bounded by N.

Proof. For each $a \in U_{\mathbb{K},n}(\epsilon)$, we are going to transform the minimal polynomial of $f(u,v)$ over $\mathbb{K}(\phi_{(u,v)})$ in a polynomial having $f(u,a)$ as a root but without increasing the degree.

Since $f(u,v)$ is algebraic over $\mathbb{K}(\phi_{(u,v)})$ and $\phi_1,\dots,\phi_m$ are algebraically independent over $\mathbb{K}$, there is $P \in \mathbb{K}[X_1,\dots,X_{2m}][Y]$ such that $P(\phi_{(u,v)};Y) \neq 0$ and $P(\phi_{(u,v)}; f(u,v)) = 0$. Hence, there exists $N \in \mathbb{N}$ such that

$$P(X_1,\dots,X_{2m};Y) = \sum_{i,\mu,\nu \leq N} c_{i,\mu,\nu}\, X_1^{\mu_1}\dots X_m^{\mu_m} X_{m+1}^{\nu_1}\dots X_{2m}^{\nu_m} Y^i,$$

with $c_{i,\mu,\nu} \in \mathbb{K}$ and $\delta \leq N$ means $\delta_1 \leq N,\dots,\delta_m \leq N$, for each $\delta \in \mathbb{N}^m$. We will prove that this N is the required in the statement of the lemma. Firstly, we prove some claims.

Claim (1) *There exists an open dense subset U of $U_{\mathbb{K},n}(\epsilon)$ such that for each $a \in U$, $P(X_1,\dots,X_m,\phi(a);Y) \in \mathbb{K}[X_1,\dots,X_m][Y]$ is a non-zero polynomial.*

Proof of Claim (1). By Lemma 1.3.(1) there exists an open dense subset $W \subset U_{\mathbb{K},n}(\epsilon)$ such that

$$W \subset \{a \in U_{\mathbb{K},n}(\epsilon) \mid \phi(a) \in \mathbb{K}^m\}$$

and ${}^{\mathrm{a}}\phi : W \to \mathbb{K}^m : a \mapsto \phi(a)$ is analytic. Let

$$U := \{a \in W \mid P(X_1,\dots,X_m,\phi(a);Y) \neq 0\}.$$

Since W is an open dense subset of $U_{\mathbb{K},n}(\epsilon)$, to prove the claim it is enough to show that $W \setminus U$ is closed and nowhere dense in W. Clearly $W \setminus U$ is closed in W because ${}^{\mathrm{a}}\phi$ is continuous in W. To prove the density, we note that if $W \setminus U$ contains an open subset of W then

$$\{a \in U_{\mathbb{K},n}(\epsilon) \mid P(\phi(u), \phi(a); Y) \in \mathscr{M}_{\mathbb{K},n+1} \text{ and } P(\phi(u), \phi(a); Y) = 0\}$$

contains an open subset of $U_{\mathbb{K},n}(\epsilon)$ and therefore $P(\phi_{(u,v)}; Y) = 0$ by Lemma 1.3.(3). This finishes the proof of Claim (1). □

Claim (2) *For each $a \in U_{\mathbb{K},n}(\epsilon)$, there exists $Q_a \in \mathbb{K}[X_1, \ldots, X_m][Y]$ such that $Q_a(\phi(u); Y)$ is not identically zero, $Q_a(\phi(u); f(u,a)) = 0$ and Q_a is a sum of monomials of the form*

$$c\, X_1^{\mu_1} \ldots X_m^{\mu_m} Y^i, \quad c \in \mathbb{K},\ \mu_1, \ldots, \mu_m \leq N,\ i \leq N.$$

Proof of Claim (2). We follow the proof of [**8**, Ch. IX. §5. Theorem 5]. For each $a \in U$, where U is as in the proof of Claim (1), let

$$P_a(X_1, \ldots, X_m; Y) = \sum_{i,\mu \leq N} b_{i,\mu,a}\ X_1^{\mu_1} \ldots X_m^{\mu_m} Y^i$$

denote the polynomial $P(X_1, \ldots, X_m, \phi(a); Y)$. We have that U is dense in $U_{\mathbb{K},n}(\epsilon)$ and $P_a \neq 0$ for all $a \in U$. For each $a \in U$, we define

$$E(P_a) := \sum_{i,\mu \leq N} \|b_{i,\mu,a}\|^2.$$

We note that $E(P_a) > 0$, for all $a \in U$. For each $a \in U$, let

$$Q_a(X_1, \ldots, X_m; Y) := \sum_{i,\mu \leq N} c_{i,\mu,a}\ X_1^{\mu_1} \ldots X_m^{\mu_m} Y^i,$$

where

$$c_{i,\mu,a} := \frac{b_{i,\mu,a}}{\sqrt{E(P_a)}}.$$

Hence, for each $a \in U$, we have that $Q_a(\phi(u); Y)$ is not identically zero, $Q_a(\phi(u); f(u,a)) = 0$ and $E(Q_a) = 1$. We define

$$\vec{v}(a) := (c_{i,\mu,a})_{i,\mu \leq N} \in \{z \in \mathbb{K}^{(N+1)^{(m+1)}} \mid \|z\| = 1\}.$$

Take $a \in U_{\mathbb{K},n}(\epsilon) \setminus U$. Since U is an open dense subset of $U_{\mathbb{K},n}(\epsilon)$, there exists a sequence $\{a_k\}_{k \in \mathbb{N}} \subset U$ that converges to a. For each a_k, the identity $Q_{a_k}(\phi(u); f(u, a_k)) = 0$ holds, therefore

$$\sum_{i,\mu \leq N} c_{i,\mu,a_k} \phi_1(u)^{\mu_1} \ldots \phi_m(u)^{\mu_m} f(u, a_k)^i = 0.$$

By hypothesis there are $\alpha, \beta \in \mathcal{O}_{\mathbb{K},2n}$, $\beta \neq 0$, convergent on $U_{2n}(\epsilon)$, such that $f(u,v) = \frac{\alpha(u,v)}{\beta(u,v)}$ and $\beta(u,a) \neq 0$ for all $a \in U_{\mathbb{K},n}(\epsilon)$. In particular

$$\sum_{i,\mu \leq N} c_{i,\mu,a_k} \phi_1(u)^{\mu_1} \ldots \phi_m(u)^{\mu_m} \alpha(u, a_k)^i \beta(u, a_k)^{N-i} = 0. \tag{1.1}$$

Since $\{z \in \mathbb{K}^{(N+1)^{(m+1)}} \mid \|z\| = 1\}$ is compact, taking a suitable subsequence we can assume that the sequence $\{\vec{v}(a_k)\}_{k \in \mathbb{N}}$ is convergent. For each $i, \mu \leq N$, we define

$$c_{i,\mu,a} := \lim_{k \to \infty} c_{i,\mu,a_k}.$$

Since α and β are continuous, when k tends to infinity equation (1.1) becomes

$$\sum_{i,\mu\leq N} c_{i,\mu,a}\phi_1(u)^{\mu_1}\dots\phi_m(u)^{\mu_m}\alpha(u,a)^i\beta(u,a)^{N-i}=0.$$

So dividing by $\beta(u,a)^N$, we also have

$$\sum_{i,\mu\leq N} c_{i,\mu,a}\phi_1(u)^{\mu_1}\dots\phi_m(u)^{\mu_m}f(u,a)^i=0$$

and hence the polynomial

$$Q_a(X_1,\dots,X_{m+1};Y):=\sum_{i,\mu\leq N} c_{i,\mu,a}X_1^{\mu_1}\dots X_m^{\mu_m}Y^i$$

satisfies $Q_a(\phi(u),f(u,a))=0$. We note that $E(Q_a)=\lim_{k\to\infty}E(Q_{a_k})=1$, so $Q_a\neq 0$. Since $\phi_1,\dots,\phi_m$ are algebraically independent over $\mathbb{K}$ and $Q_a(X_1,\dots,X_m,Y)\neq 0$, we have $Q_a(\phi(u),Y)$ is not identically zero. This finishes the proof of Claim (2). □

Claim (2) implies that $f(u,a)$ is algebraic over $\mathbb{K}(\phi(u))$, for all $a\in U_{\mathbb{K},n}(\epsilon)$. It remains to check the conditions on N and on the minimal polynomials. Fix $a\in U_{\mathbb{K},n}(\epsilon)$. Let $A(Y):=Q_a(\phi(u);Y)$, where Q_a is the polynomial of Claim (2). By definition of Q_a, we have

$$A(Y)=A_{\ell_1}Y^{\ell_1}+\sum_{i=0}^{\ell_1-1}A_iY^i,\quad A_0,\dots,A_{\ell_1}\in\mathbb{K}[\phi(u)],\ A_{\ell_1}\neq 0,\ \ell_1\leq N$$

where each of A_0,…,A_{ℓ_1} is a sum of monomials of the form

$$c\,\phi_1(u)^{\mu_1}\dots\phi_m(u)^{\mu_m},\quad c\in\mathbb{K},\ 0\leq\mu_1,\dots\mu_m\leq N.$$

Denote

$$B(Y)=Y^\ell+\frac{\sum_{i=0}^{\ell-1}B_iY^i}{B_\ell},\quad B_0,\dots,B_\ell\in\mathbb{K}[\phi(u)]$$

the minimal polynomial of $f(u,a)$ over $\mathbb{K}(\phi(u))$. Since $f(u,a)$ is a root of both $A(Y)$ and $B(Y)$, $\ell\leq\ell_1\leq N$ and there exists

$$C(Y)=C_{\ell_2}Y^{\ell_2}+\sum_{i=0}^{\ell_2-1}C_iY^i,\quad C_0,\dots,C_{\ell_2}\in\mathbb{K}[\phi(u)]$$

such that $A(Y)=B(Y)\,C(Y)$. Therefore

$$B_\ell\left(A_{\ell_1}Y^{\ell_1}+\sum_{i=0}^{\ell_1-1}A_iY^i\right)=\left(B_\ell Y^\ell+\sum_{i=0}^{\ell-1}B_iY^i\right)\left(C_{\ell_2}Y^{\ell_2}+\sum_{i=0}^{\ell_2-1}C_iY^i\right).$$

We note that $\mathbb{K}[\phi(u)]\cong\mathbb{K}[X_1,\dots,X_m]$ because $\phi_1,\dots,\phi_m$ are algebraically independent over $\mathbb{K}$. Since $\mathbb{K}[\phi(u)][Y]$ is an UFD and $B(Y)$ is irreducible,

$$B_\ell Y^\ell+\sum_{i=0}^{\ell-1}B_iY^i\quad\text{divides}\quad A_{\ell_1}Y^{\ell_1}+\sum_{i=0}^{\ell_1-1}A_iY^i\quad\text{in }\mathbb{K}[\phi(u)][Y].$$

This implies that each polynomial B_0,…,B_ℓ is a sum of monomials of the form

$$c\,\phi_1(u)^{\mu_1}\dots\phi_m(u)^{\mu_m},\quad c\in\mathbb{K},\ 0\leq\mu_1,\dots\mu_m\leq N.$$

This proves the statement for $f(u,a)$. Since a was fixed, we are done. □

Although we will not use it here, we note that the previous proof can be adapted to show that if $\phi \in \mathscr{M}_{\mathbb{K},n}^n$ admits an AAT then the formal derivatives $\partial_{u_j}\phi_i$ are algebraic over $\mathbb{K}(\phi)$, for each $i, j \in \{1, \ldots, n\}$.

Corollary 1.5. *Let $\epsilon > 0$ and let $\phi \in \mathscr{M}_{\mathbb{K},n}^n$ be convergent on $U_n(\epsilon)$. If ϕ admits an AAT then ϕ_{u+a} is algebraic over $\mathbb{K}(\phi)$, for each $a \in U_{\mathbb{K},n}(\epsilon)$.*

Proof. Let $f(u,v) := \phi(u+v)$. Since ϕ admits an AAT and $\phi_{u+a} = f(u,a) \in \mathscr{M}_{\mathbb{K},n}$, for all $a \in U_n(\epsilon)$, we are under the hypothesis of Lemma 1.4. $\square$

Next lemma is an adaptation of [**45**, Ch. 5. §13. Theorem 1] from the context of abelian functions to our setting.

Lemma 1.6. *Let $\epsilon > 0$ and let $\Psi := (\psi_0, \ldots, \psi_n) \in \mathscr{M}_{\mathbb{K},n}^{n+1}$ be convergent on $U_n(\epsilon)$. Assume $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$ and ψ_0 is algebraic over $\mathbb{K}(\psi_1, \ldots, \psi_n)$. Denote $\psi := (\psi_1, \ldots, \psi_n)$ and let ℓ be the degree of the minimal polynomial of ψ_0 over $\mathbb{K}(\psi)$. Pick $f(u,v) \in \mathscr{M}_{\mathbb{K},2n}$ be convergent on $U_{2n}(\epsilon)$. Suppose that there exists $N \in \mathbb{N}$ such that for each $a \in U_{\mathbb{K},n}(\epsilon)$, it holds that $f(a,v) \in \mathbb{K}(\Psi(v))$ and there are $R_i, S_i \in \mathbb{K}[X_1, \ldots, X_n]^{\leq N}$, $S_i \neq 0$, satisfying*

$$f(u,a) = \sum_{i=0}^{\ell-1} \frac{R_i(\psi(u))}{S_i(\psi(u))} \psi_0(u)^i.$$

Then, $f(u,v) \in \mathbb{K}(\Psi_{(u,v)})$.

Proof. We denote by $H_1(u,v), \ldots, H_m(u,v)$ the monomials

$$\psi_1(u)^{\alpha_1} \ldots \psi_n(u)^{\alpha_n} f(u,v), \quad \psi_0(u)^{\alpha_0}\psi_1(u)^{\alpha_1} \ldots \psi_n(u)^{\alpha_n},$$

with $0 \leq \alpha_0 \leq \ell - 1$ and $0 \leq \alpha_1, \ldots, \alpha_n \leq \ell N$ (so $m := (\ell+1)(\ell N + 1)^n$).

Firstly we show that if there exists an equation of the form

$$\xi_1(v)H_1(u,v) + \ldots + \xi_m(v)H_m(u,v) = 0, \tag{1.2}$$

where $\xi_1(v), \ldots, \xi_m(v) \in \mathbb{K}(\Psi(v))$ and not all of them are 0 then we are done. We may assume that there exists $k > 0$ such that $f(u,v)$ appears in the monomial $H_i(u,v)$ if and only if $i \leq k$. If $\xi_i(v) \neq 0$, for some $i \leq k$, then we are done.

Indeed, since $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$, solving equation (1.2) with respect to $f(u,v)$ we deduce that $f(u,v) \in \mathbb{K}(\Psi_{(u,v)})$. So it is enough to show that $\xi_i(v) \neq 0$ for some $i \leq k$. Suppose for a contradiction that $\xi_i(v) = 0$ for all $i \leq k$. Since not all $\xi_1(v), \ldots, \xi_m(v)$ are 0 we may assume that $\xi_{k+1}(v) \neq 0$. By Lemma 1.3.(1) there exists $a \in U_{\mathbb{K},n}(\epsilon)$ such that $\xi_1(a), \ldots, \xi_m(a) \in \mathbb{K}$ and $\xi_{k+1}(a) \neq 0$. Observe that $H_{k+1}(u,v), \ldots, H_m(u,v) \in \mathbb{K}(\Psi(u))$. As they do not depend on v, we write $H_i(u)$. Evaluating equation (1.2) at $v = a$ we obtain that

$$\xi_{k+1}(a)H_{k+1}(u) + \ldots + \xi_m(a)H_m(u) = 0$$

where $\xi_{k+1}(a) \neq 0$. Since the degree of each $H_i(u)$ in the variable $\psi_0(u)$ is smaller than that of the minimal polynomial of ψ_0 over $\mathbb{K}(\psi)$, we must have $\xi_{k+1}(a) = \ldots = \xi_m(a) = 0$, a contradiction.

We now show how to obtain equation (1.2). If $f(u,a) = 0$, for each $a \in U_{\mathbb{K},n}(\epsilon)$, then $f(u,v) = 0$ by Lemma 1.3.(3) and there is nothing to prove. So we may assume that there exists $a \in U_{\mathbb{K},n}(\epsilon)$ such that $f(u,a) \neq 0$. By hypothesis for this a there are $R_i, S_i \in \mathbb{K}[X_1,\ldots,X_n]^{\leq N}$, $S_i \neq 0$, such that

$$f(u,a) = \sum_{i=0}^{\ell-1} \frac{R_i(\psi(u))}{S_i(\psi(u))}\psi_0(u)^i.$$

Clearing denominators we get

$$S(\psi(u))f(u,a) = \sum_{i=0}^{\ell-1} R_i'(\psi(u))\psi_0(u)^i \tag{1.3}$$

where $R_i', S \in \mathbb{K}[X_1,\ldots,X_n]^{\leq \ell N}$ and $S \neq 0$. We also recall that ℓ and N do not depend on a. Now we follow [**8**, Ch. IX. §5. Lemma 6]. Let $u_{(1)},\ldots,u_{(m)}$ be independent n-tuples of variables and let $D(v,u_{(1)},\ldots,u_{(m)})$ be the determinant of

$$H(v,u_{(1)},\ldots,u_{(m)}) := \begin{bmatrix} H_1(u_{(1)},v) & H_1(u_{(2)},v) & \ldots & H_1(u_{(m)},v) \\ H_2(u_{(1)},v) & H_2(u_{(2)},v) & \ldots & H_2(u_{(m)},v) \\ \vdots & \vdots & \ddots & \vdots \\ H_m(u_{(1)},v) & H_m(u_{(2)},v) & \ldots & H_m(u_{(m)},v) \end{bmatrix}.$$

By equation (1.3), for each $a \in U_{\mathbb{K},n}(\epsilon)$, the monomials $H_1(u,a),\ldots,H_m(u,a)$ are linearly dependent over $\mathbb{K}$. Since

$$\{a \in U_{\mathbb{K},n}(\epsilon) \mid D(a,u_{(1)},\ldots,u_{(m)}) \in \mathscr{M}_{\mathbb{K},mn} \text{ and } D(a,u_{(1)},\ldots,u_{(m)}) = 0\}$$

is $U_{\mathbb{K},n}(\epsilon)$, $D = 0$ by Lemma 1.3.(3). Expanding the determinant of H with respect to its last column, replacing $u_{(m)}$ by u and denoting $(u_{(1)},\ldots,u_{(m-1)})$ by $u_{(*)}$, we obtain a new equation of the form

$$\chi_1(v,u_{(*)})H_1(u,v) + \ldots + \chi_m(v,u_{(*)})H_m(u,v) = 0,$$

where

$$\chi_1(v,u_{(*)}),\ldots,\chi_m(v,u_{(*)}) \in \mathbb{K}\big(H_j(u_{(i)},v) \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m\big).$$

Without loss of generality we may assume that not all the $\chi_1,\ldots,\chi_m$ are 0.

Indeed, there is a minor of D of order $\nu \in (0,m)$ that is not zero and thus we can assume that $\nu = m-1$. Now, fix $i \in \{1,\ldots,m\}$ such that $\chi_i(v,u_{(*)}) \neq 0$. Then by Lemma 1.3.(2) there exists $b := (b_{(1)},\ldots,b_{(m-1)}) \in U_{\mathbb{K},(m-1)n}(\epsilon)$ such that

$$\chi_1(v,b),\ldots,\chi_m(v,b) \in \mathscr{M}_{\mathbb{K},n} \text{ and } \chi_i(v,b) \neq 0.$$

We note that by hypothesis $f(b_{(1)},v),\ldots,f(b_{(m-1)},v) \in \mathbb{K}(\Psi(v))$, therefore $\chi_1(v,b),\ldots,\chi_m(v,b) \in \mathbb{K}(\Psi(v))$. Since $\chi_i(v,b) \neq 0$, evaluating $u_{(*)}$ at b we obtain an equation as in (1.2). This concludes the proof. □

With the previous lemmas we can now follow the proof of [**45**, Ch. 5. §13. Theorem 1] and apply it to our context.

Proof of Fact 1.2. Let $\psi := (\psi_1,\ldots,\psi_n)$. Let

$$P(X) = X^\ell + P_1X^{\ell-1} + \ldots + P_\ell \in \mathbb{K}(\psi)[X].$$

be the minimal polynomial of ψ_0 over $\mathbb{K}(\psi)$. We note that $\mathbb{K}(\Psi)$ is isomorphic to $\mathbb{K}(\psi)[X]/(P(X))$. For each $a \in U_{\mathbb{K},n}(\epsilon)$, we write $f_a = f(u,a)$. As $f_a \in \mathbb{K}(\Psi)$ and $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$, we have

$$f_a = S_{a,1}(\psi)\psi_0^{\ell-1} + S_{a,2}(\psi)\psi_0^{\ell-2} + \ldots + S_{a,\ell-1}(\psi)\psi_0 + S_{a,\ell}(\psi)$$

for some $S_{a,1}, \ldots, S_{a,\ell} \in \mathbb{K}(X_1, \ldots, X_n)$. By Lemma 1.6 we only need to check that there exists $N \in \mathbb{N}$ such that for each $a \in U_{\mathbb{K},n}(\epsilon)$, each $S_{a,i}$ is the quotient of two polynomials in $\mathbb{K}[X_1, \ldots, X_n]^{\leq N}$.

For each α algebraic over $\mathbb{K}(\psi)$, let $\sigma(\alpha)$ denote its trace. Fix $a \in U_{\mathbb{K},n}(\epsilon)$ and let $\xi_1, \ldots, \xi_\ell$ be the ℓ roots of $P(X)$. We have $\sigma(\psi_0) = \xi_1 + \ldots + \xi_\ell \in \mathbb{K}(\psi)$. For each $i \in \{1, \ldots, \ell\}$, we define

$$f_a^{(i)} := S_{a,1}(\psi)\xi_i^{\ell-1} + S_{a,2}(\psi)\xi_i^{\ell-2} + \ldots + S_{a,\ell-1}(\psi)\xi_i + S_{a,\ell}(\psi).$$

Let

$$L := \begin{bmatrix} \xi_1^{\ell-1} & \xi_2^{\ell-1} & \ldots & \xi_\ell^{\ell-1} \\ \xi_1^{\ell-2} & \xi_2^{\ell-2} & \ldots & \xi_\ell^{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1 & \xi_2 & \ldots & \xi_\ell \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Observe that $LL^{\mathrm{t}} = \left[\sigma(\psi_0^{2\ell-i-j})\right]$, so the coefficients of LL^{t} belong to $\mathbb{K}(\psi)$. Since $\det(LL^{\mathrm{t}}) = \prod_{1\leq i<j\leq \ell}(\xi_i - \xi_j)^2$ and $\mathbb{K}(\psi)$ is separable, we have $\det(LL^{\mathrm{t}}) \neq 0$, so LL^{t} is invertible. We note that $\mathbb{K}(\psi)$ is isomorphic to $\mathbb{K}(X_1, \ldots, X_n)$ because $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$. Hence, we identify each $S_{a,i}$ with $S_{a,i}(\psi)$. With this convention,

$$\left[f_a^{(1)}, f_a^{(2)}, \ldots, f_a^{(\ell)}\right] = \left[S_{a,1}, S_{a,2}, \ldots, S_{a,\ell}\right] L$$

and $\sigma(f_a\psi_0^j) = \sum_{i=1}^{\ell} f_a^{(i)}\xi_i^j$, for each $j \in \mathbb{N}$, so

$$\left[S_{a,1}, S_{a,2}, \ldots, S_{a,\ell}\right] = \left[\sigma(f_a\psi_0^{\ell-1}), \sigma(f_a\psi_0^{\ell-2}), \ldots, \sigma(f_a\psi_0), \sigma(f_a)\right](LL^{\mathrm{t}})^{-1}.$$

Since L does not depend on a, it is enough to show that there exists $N \in \mathbb{N}$ such that for each $a \in U_{\mathbb{K},n}(\epsilon)$, each $\sigma(f_a\psi_0^{\ell-1}), \ldots, \sigma(f_a)$ can be written in the form $A(\psi)/B(\psi)$, for some $A, B \in \mathbb{K}[X_1, \ldots, X_n]^{\leq N}$ and $B \neq 0$.

Indeed, fix $j \in \{0, \ldots, \ell-1\}$. By hypothesis both $f(u,v)$ and $\psi_0(u)$ are algebraic over $\mathbb{K}(\psi_{(u,v)})$. Thus, by Lemma 1.4 applied to $f(u,v)\psi_0(u)^j$ we deduce that there exists $N_j \in \mathbb{N}$ such that for every $a \in U_{\mathbb{K},n}(\epsilon)$, the minimal polynomial of $f_a\psi_0^j$ over $\mathbb{K}(\psi)$ can be written in the form

$$Y^\ell + \sum_{i=0}^{\ell-1}\frac{A_i(\psi)}{B_i(\psi)}Y^i,$$

for some $A_i, B_i \in \mathbb{K}[X_1, \ldots, X_n]^{\leq N_j}$ and $B_i \neq 0$. In particular, $\sigma(f_a\psi_0^j) = -A_{\ell-1}(\psi)/B_{\ell-1}(\psi)$. Finally, it is enough to take N the maximum between of $N_0, \ldots, N_{\ell-1}$. $\square$

3. Some results on AAT and the proof of the Extension Theorem

We show additional properties for those elements of $\mathscr{M}^n_{\mathbb{K},n}$ that admit an AAT.

LEMMA 1.7. *Let $\phi, \psi \in \mathscr{M}^n_{\mathbb{K},n}$ and suppose that ϕ is algebraic over $\mathbb{K}(\psi)$. If ϕ admits an AAT then ψ admits an AAT. The converse is also true, provided $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$.*

PROOF. Assume that ϕ admits an AAT, hence $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$ because ϕ is algebraic over $\mathbb{K}(\psi)$. To check that ψ_{u+v} is algebraic over $\mathbb{K}(\psi_{(u,v)})$ it is enough to show that ψ_{u+v} is algebraic over $\mathbb{K}(\phi_{u+v})$, ϕ_{u+v} is algebraic over $\mathbb{K}(\phi_{(u,v)})$ and $\phi_{(u,v)}$ is algebraic over $\mathbb{K}(\psi_{(u,v)})$. The three conditions above are trivially satisfied because ϕ admits an AAT and both ϕ is algebraic over $\mathbb{K}(\psi)$ and ψ is algebraic over $\mathbb{K}(\phi)$. The converse follows by symmetry because if $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$ then ψ is algebraic over $\mathbb{K}(\phi)$. $\square$

Now, we adapt to our context a result on AAT due to H.A.Schwarz, see [**16**, Ch. XXI. Art. 389] for details.

LEMMA 1.8. *Let $\epsilon > 0$ and let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ such that it admits an AAT. Then, there exist a finite subset $\mathcal{C} \subset U_{\mathbb{K},n}(\epsilon)$, with $0 \in \mathcal{C}$ and $\mathcal{C} = -\mathcal{C}$, and $\epsilon' \in (0, \epsilon]$ satisfying: each element of $\mathbb{K}(\phi_{u+a} \mid a \in \mathcal{C})$ is convergent on $U_n(2\epsilon')$, and there exist $A_0, \ldots, A_N \in \mathbb{K}(\phi_{(u+a,v+a)} \mid a \in \mathcal{C})$ convergent on $U_{2n}(2\epsilon')$ such that ϕ_{u+v} is algebraic over $\mathbb{K}(A_0, \ldots, A_N)$ and, for each $j \in \{0, \ldots, N\}$,*

$$A_j(u, v) = A_j(u + a, v - a), \text{ for all } a \in U_{\mathbb{K},n}(\epsilon'). \tag{1.4}$$

PROOF. Fix $i \in \{1, \ldots, n\}$. Let $\mathcal{S}_0 := \{0\}$ and $\mathbb{K}_0 := \mathbb{K}(\phi_{(u,v)})$. Let

$$P_0(X) = X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}(u, v)X^j$$

be the minimal polynomial of $\phi_i(u+v)$ over $\mathbb{K}_0$. If each $A_{0,j}$ satisfies equation (1.4) for $\epsilon' = 2^{-1}\epsilon$ then we are done for this i letting $\epsilon' := 2^{-1}\epsilon$, $\mathcal{C} := \mathcal{S}_0$ and $A_j := A_{0,j}$, for each $0 \le j \le \ell_0$. Otherwise, there exists $a_1 \in U_{\mathbb{K},n}(2^{-1}\epsilon)$ such that

$$Q_0(X) := X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}(u, v)X^j - X^{\ell_0+1} - \sum_{j=0}^{\ell_0} A_{0,j}(u + a_1, v - a_1)X^j$$

is not zero. Since $u + v = (u + a_1) + (v - a_1)$, we deduce that $\phi_i(u + v)$ is a root of $Q_0(X)$. Let $\mathcal{S}_1 := \mathcal{S}_0 \cup \{a_1, -a_1\}$ and $\mathbb{K}_1 := \mathbb{K}(\phi_{u+a,v+a} \mid a \in \mathcal{S}_1)$. By definition $\mathbb{K}_0 \subset \mathbb{K}_1$. Let

$$P_1(X) = X^{\ell_1+1} + \sum_{j=0}^{\ell_1} A_{1,j}(u, v)X^j$$

be the minimal polynomial of $\phi_i(u + v)$ over $\mathbb{K}_1$. We note that the elements of $\mathbb{K}_1$ are convergent on $U_{2n}(2^{-1}\epsilon)$. If each $A_{1,j}$ satisfies equation (1.4) for $\epsilon' = 2^{-2}\epsilon$ then we are done for this i letting $\epsilon' := 2^{-2}\epsilon$, $\mathcal{C} := \mathcal{S}_1$ and $A_j := A_{1,j}$, for each $0 \le j \le \ell_1$. Otherwise, we can repeat the process to obtain sets $\mathcal{S}_2$, $\mathcal{S}_3$ and so on, where the set $\mathcal{S}_k$ is obtained from the set $\mathcal{S}_{k-1}$ as

$$\mathcal{S}_k := \mathcal{S}_{k-1} \cup \{a + a_k \mid a \in \mathcal{S}_{k-1}\} \cup \{a - a_k \mid a \in \mathcal{S}_{k-1}\},$$

for some $a_k \in U_{\mathbb{K},n}(2^{-k}\epsilon)$ such that Q_{k-1} is not 0. Similarly, we obtain $\mathbb{K}_k := \mathbb{K}(\phi_{u+a,v+a} \mid a \in \mathcal{S}_k)$ whose elements are convergent on $U_{2n}(2^{-k}\epsilon)$. Since in the k repetition the degree of P_k is smaller than that of P_{k-1}, this process eventually stops, say at step s. Letting $\epsilon' := 2^{-s-1}\epsilon$, $\mathcal{C} := \mathcal{S}_s$ and $A_j := A_{s,j}$, for each $0 \le j \le \ell_s$, we are done for this i. The elements $A_0, \ldots, A_{\ell_s}$ are convergent on $U_{2n}(2\epsilon')$ because they are elements of $\mathbb{K}_s$.

For each i, $1 \le i \le n$, denote by ϵ'_i, $\mathcal{C}_i$ and $A^i_0, \ldots, A^i_{N_i}$ the elements ϵ', $\mathcal{C}$ and $A_1, \ldots, A_{\ell_s}$ previously obtained for that choice of i. To complete the proof, take $\mathcal{C} := \bigcup_i \mathcal{C}_i$, $\epsilon' := \min_i\{\epsilon'_i\}$, and let $\{A_0, \ldots, A_N\}$ be the union of the sets $\{A^i_0, \ldots, A^i_{N_i}\}$. □

We need two additional lemmas before proving Theorem 1.11.

LEMMA 1.9. *Let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ admit an AAT. Then, $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$.*

PROOF. Take $\epsilon > 0$ such that $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ is convergent on $U_n(\epsilon)$. Since ϕ admits an AAT, we know that $\phi(u+v)$ is algebraic over $\mathbb{K}(\phi(u), \phi(v))$. Taking into account transcendence degrees, it follows that $\phi(v)$ is algebraic over $\mathbb{K}(\phi(u+v), \phi(u))$. For some $a \in U_{\mathbb{K},n}(\epsilon)$, we may substitute v by $-u+a$, so $\phi(-u+a)$ is algebraic over $\mathbb{K}(\phi(u))$. By Corollary 1.5, $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(-u+a))$ and hence over $\mathbb{K}(\phi(u))$. □

LEMMA 1.10. *Let $\epsilon > 0$. Let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ such that it admits an AAT. Then there exist $\epsilon_1 \in (0, \epsilon]$ and $\Psi := (\psi_0, \ldots, \psi_n) \in \mathscr{M}^{n+1}_{\mathbb{K},n}$ convergent on $U_n(\epsilon_1)$ and algebraic over $\mathbb{K}(\phi)$ satisfying $\psi := (\psi_1, \ldots, \psi_n)$ admits an AAT, ψ_0 is algebraic over $\mathbb{K}(\psi)$ and, for each $f \in \mathbb{K}(\Psi)$, $f(-u) \in \mathbb{K}(\Psi(u))$, there exists $\delta \in (0, \epsilon_1]$ such that for each $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{K}(\Psi)$ and f_{u+a} is convergent on $U_n(\epsilon_1)$.*

PROOF. We will define a field $\mathbb{L}$ generated over $\mathbb{K}$ by certain elements of $\mathscr{M}_{\mathbb{K},n}$, next we will prove that each $f \in \mathbb{L}$ satisfy the conclusion of the lemma and finally we find a primitive element Ψ such that $\mathbb{L} = \mathbb{K}(\Psi)$.

Let $\epsilon' \in (0, \epsilon]$, $\mathcal{C} \subset U_{\mathbb{K},n}(\epsilon)$ and $A_0, \ldots, A_N \in \mathbb{K}(\phi_{(u+c,v+c)} \mid c \in \mathcal{C})$ be the ones provided by Lemma 1.8 for ϕ. By Lemma 1.3.(1) there exists an open dense subset $U \subset U_{\mathbb{K},n}(\epsilon')$ such that

$$U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid \phi(a+c) \in \mathbb{K}^n \text{ for all } c \in \mathcal{C}\}$$

and

$$U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid A_0(u,a), \ldots, A_N(u,a) \in \mathscr{M}_{\mathbb{K},n}\}.$$

In particular, $U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid \phi(a) \in \mathbb{K}^n\}$ because $0 \in \mathcal{C}$. Since U is open there exist $b \in U$ and $\epsilon'' \in (0, \epsilon' - \|b\|]$ such that

$$V := \{a \in U_{\mathbb{K},n}(\epsilon') \mid \|a - b\| < \epsilon''\} \subset U.$$

Fix such b. Then, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, each $A_j(u, a+b)$, $j = 1, \ldots, N$ is an element of $\mathscr{M}_{\mathbb{K},n}$. We note that since each $A_j(u,v)$ is convergent on $U_{2n}(2\epsilon')$

and by definition of b and ϵ'', each $A_j(u, a+b)$ is convergent on $U_n(\epsilon')$, for each $a \in U_{\mathbb{K},n}(\epsilon'')$. Also, since each A_j satisfies the equation (1.4) of Lemma 1.8,

$$A_j(u, a+b) = A_j(u+a, b) \text{ for all } a \in U_{\mathbb{K},n}(\epsilon''). \tag{1.5}$$

For each $j \in \{0, \ldots, N\}$, we define $B_j(u) := A_j(u, b)$. Let

$$\mathbb{L}_1 := \mathbb{K}((B_j)_{u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N).$$

Since, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, each $A_j(u, a+b)$ is convergent on $U_n(\epsilon')$, by equation (1.5) all the elements of $\mathbb{L}_1$ are convergent on $U_n(\epsilon')$ and in particular in $U_n(\epsilon'')$. Let

$$\mathbb{L}_2 := \mathbb{K}((B_j)_{-u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N).$$

Note that all the elements of $\mathbb{L}_2$ are also convergent on $U_n(\epsilon'')$. Hence, if we define

$$\mathbb{L} := \mathbb{K}((B_j)_{u+a}, (B_j)_{-u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N),$$

all the elements of $\mathbb{L}$ are also convergent on $U_n(\epsilon'')$.

Let us show that

$$\mathbb{L} \subset \mathbb{K}(\phi_{u+c}, \phi_{-u+c} \mid c \in \mathcal{C})$$

and that each element of $\mathbb{L}$ is algebraic over $\mathbb{K}(\phi)$.

We begin proving that

$$\mathbb{L}_1 \subset \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$$

and that each element of $\mathbb{L}_1$ is algebraic over $\mathbb{K}(\phi)$. Fix $j \in \{0, \ldots, N\}$ and $a \in U_{\mathbb{K},n}(\epsilon'')$. We recall from Lemma 1.8 that $A_j(u, v)$ is convergent on $U_{2n}(2\epsilon')$ and $A(u, v) \in \mathbb{K}(\phi_{(u+c,v+c)} \mid c \in \mathcal{C})$. Hence we can evaluate $A_j(u, v)$ at $v = a+b$ to deduce that $A_j(u, a+b) \in \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$. Thus, by equation (1.5), $A_j(u+a, b) \in \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$. Hence, $\mathbb{L}_1 \subset \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$ and therefore, by Corollary 1.5, each element of $\mathbb{L}_1$ is algebraic over $\mathbb{K}(\phi)$. By symmetry of $\mathcal{C}$, $\mathbb{L}_2 \subset \mathbb{K}(\phi_{-u+c} \mid c \in \mathcal{C})$ and each element of $\mathbb{L}_2$ is algebraic over $\mathbb{K}(\phi(-u))$. Therefore $\mathbb{L} \subset \mathbb{K}(\phi_{u+c,-u+c} \mid c \in \mathcal{C})$ and, since by Lemma 1.9 we have that $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$, we deduce that each element of $\mathbb{L}$ is algebraic over $\mathbb{K}(\phi(u))$, as required.

Next, we show that $\phi_1(u+b), \ldots, \phi_n(u+b)$ are algebraically independent over $\mathbb{K}$. Let $P \in \mathbb{K}[X_1, \ldots, X_n]$ be such that $P(\phi_{u+b}) = 0$. By notation, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, we have that $P(\phi_{u+b}(a)) = 0$ if and only if $P(\phi(a+b)) = 0$. Hence,

$$V \subset \{a \in U_{\mathbb{K},n}(\epsilon) \mid \phi(a) \in \mathbb{K} \text{ and } P(\phi(a)) = 0\}.$$

Since V is open in $U_{\mathbb{K},n}(\epsilon)$, $P(\phi) = 0$ by the identity principle. Since $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$, $P = 0$ and we are done.

Next, we show that $\mathbb{L}$ is finitely generated over $\mathbb{K}$ and its transcendence degree is n. Firstly, we note that ϕ is algebraic over $\mathbb{K}(\phi_{u+b})$ because the coordinate functions of ϕ_{u+b} are algebraically independent over $\mathbb{K}$ and ϕ_{u+b} is algebraic over $\mathbb{K}(\phi)$ by Corollary 1.5. Since ϕ_{u+v} is algebraic over $\mathbb{K}(A_0, \ldots, A_N)$, evaluating each $A_j(u, v)$ at $v = b$ we deduce that ϕ_{u+b} is algebraic over $\mathbb{K}(B_0, \ldots, B_N)$. Therefore, ϕ is algebraic over $\mathbb{K}(B_0, \ldots, B_N)$. On the other hand, $\mathbb{K}(B_0, \ldots, B_N)$ is a subset of $\mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$ and the latter field is algebraic over $\mathbb{K}(\phi)$ by Corollary 1.5. Hence the three fields have transcendence degree n over $\mathbb{K}$. Recall that $\mathcal{C} = -\mathcal{C}$, so $\mathbb{K}(\phi_{-u+c} \mid c \in \mathcal{C}) = \mathbb{K}(\phi_{-u-c} \mid c \in \mathcal{C})$. We also note that $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$, so $\mathbb{K}(\phi_{u+c}, \phi_{-u-c} \mid c \in \mathcal{C})$ has transcendence degree n over $\mathbb{K}$. Now, $\mathcal{C}$ is finite and

$$\mathbb{K}(B_0(u), \ldots, B_N(u)) \subset \mathbb{L} \subset \mathbb{K}(\phi_{u+c}, \phi_{-u-c} \mid c \in \mathcal{C}),$$

therefore, $\mathbb{L}$ is finitely generated over $\mathbb{K}$ and its transcendence degree is n.

Fix $f \in \mathbb{L}$ and let us check that $f(-u) \in \mathbb{L}$ and that there exists $\delta > 0$ such that for every $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{L}$ and f_{u+a} is convergent on $U_n(\epsilon'')$.

Since $f \in \mathbb{L}$, there exist $m, m' \in \mathbb{N}$, $j(1), \ldots, j(m+m') \in \{0, \ldots, N\}$ and $a_1, \ldots, a_{m+m'} \in U_{\mathbb{K},n}(\epsilon'')$ such that f is a rational function of

$$(B_{j(1)})_{u+a_1}, \ldots, (B_{j(m)})_{u+a_m}, (B_{j(m+1)})_{-u+a_{m+1}}, \ldots, (B_{j(m+m')})_{-u+a_{m+m'}}.$$

In particular, $f(-u)$ is a rational function of

$$(B_{j(1)})_{-u+a_1}, \ldots, (B_{j(m)})_{-u+a_m}, (B_{j(m+1)})_{u+a_{m+1}}, \ldots, (B_{j(m+m')})_{u+a_{m+m'}},$$

so $f(-u) \in \mathbb{L}$. Take $\delta > 0$ such that $\delta < \epsilon'' - \max\{\|a_1\|, \ldots, \|a_{m+m'}\|\}$. Then, for all $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{L}$ and f_{u+a} is convergent on $U_n(\epsilon'')$.

Finally, take $\psi_1, \ldots, \psi_n \in \mathbb{L}$ algebraically independent over $\mathbb{K}$ and ψ_0 algebraic over $\mathbb{K}(\psi_1, \ldots, \psi_n)$ such that $\mathbb{L} = \mathbb{K}(\psi_0, \psi_1, \ldots, \psi_n)$. Now, since all the elements of $\mathbb{L}$ are algebraic over $\mathbb{K}(\phi)$, $\psi := (\psi_1, \ldots, \psi_n)$ admits an AAT by Lemma 1.7. $\square$

We now have all the ingredients to prove our main result.

THEOREM 1.11. *Let $\phi := (\phi_1, \ldots, \phi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admit an AAT. Then, there exist $\psi := (\psi_1, \ldots, \psi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT and is algebraic over $\mathbb{K}(\phi)$, and an additional meromorphic series $\psi_0 \in \mathcal{M}_{\mathbb{K},n}$ algebraic over $\mathbb{K}(\psi)$ such that:*

1. *For each $f(u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$,*
   - *(a) $f(u+v) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u), \psi_0(v), \ldots, \psi_n(v))$ and*
   - *(b) $f(-u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$.*
2. *Each $\psi_0, \ldots, \psi_n$ is the quotient of two convergent power series whose complex domain of convergence is $\mathbb{C}^n$.*

PROOF. Let $\phi := (\phi_1, \ldots, \phi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admit an AAT. Take $\epsilon > 0$ such that ϕ is convergent on $U_n(\epsilon)$. Applying Lemma 1.10 we obtain $\epsilon_1 \in (0, \epsilon]$ and $\Psi := (\psi_0, \ldots, \psi_n) \in \mathcal{M}^{n+1}_{\mathbb{K},n}$ as in the lemma. We next check that this Ψ satisfies the conditions of the theorem.

(1) By Lemma 1.10, if $f \in \mathbb{K}(\Psi)$ then $f(-u) \in \mathbb{K}(\Psi)$, so we only have to check $f(u+v) \in \mathbb{K}(\Psi_{(u,v)})$. Fix a non-constant $f \in \mathbb{K}(\Psi)$ and $\delta \in (0, \epsilon_1]$ such that $f_{u+a} \in \mathbb{K}(\Psi)$, for each $a \in U_n(\delta)$, as in Lemma 1.10. Let $0 < \varepsilon < \delta$ be such that f_{u+v} is convergent on $U_{2n}(\varepsilon)$. It is enough to show that $f_{u+v} \in \mathcal{M}_{\mathbb{K},2n}$ is algebraic over $\mathbb{K}(\Psi_{(u,v)})$ because then we can apply Fact 1.2 as $f_{u+a} \in \mathbb{K}(\Psi(u))$ and $f_{v+a} \in \mathbb{K}(\Psi(v))$, for each $a \in U_{\mathbb{K},n}(\varepsilon)$. Take $g_2, \ldots, g_n \in \mathbb{K}(\psi)$ such that $f, g_2, \ldots, g_n$ are algebraically independent over $\mathbb{K}$. Let $g := (f, g_2, \ldots, g_n)$ and we note that g is algebraic over $\mathbb{K}(\psi)$. Since ψ admits an AAT, also g admits an AAT by Lemma 1.7. Hence g_{u+v} is algebraic over $\mathbb{K}(g_{(u,v)})$ and therefore over $\mathbb{K}(\Psi_{(u,v)})$. This concludes the proof of (1).

(2) We may assume that $\psi_0 \neq 0$. Fix $i \in \{0, \dots, n\}$. We have already shown that $\psi_i(u+v) \in \mathbb{K}(\Psi_{(u,v)})$. Let $A(u,v) := \psi_i(u+v)$. By Lemma 1.10 and taking a smaller $\epsilon > 0$ if necessary, we may assume that Ψ is convergent on $U_n(\epsilon)$ and $\mathbb{K}(\Psi_{u+a}) \subset \mathbb{K}(\Psi)$, for each $a \in U_{\mathbb{K},n}(\epsilon)$. Let us show that there exists $c \in U_{\mathbb{K},n}(\epsilon)$ such that

$$A(u+c, u-c) \in \mathcal{M}_{\mathbb{K},n}.$$

Take $\alpha, \beta \in \mathcal{O}_{\mathbb{K},2n}$, $\beta \neq 0$ such that $A(u,v) = \frac{\alpha(u,v)}{\beta(u,v)}$. Suppose by contradiction that $\beta(u+c, u-c) = 0$ for all $c \in U_{\mathbb{K},n}(\epsilon)$. Then

$$\beta\left(\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}\right) = 0,$$

for all $a, b \in U_{\mathbb{K},n}(\epsilon/2)$. So $\beta(a,b) = 0$, for all $(a,b) \in U_{\mathbb{K},n}(\epsilon/2)$, that is, $\beta = 0$, which is a contradiction. Consequently,

$$\psi_i(2u) = A(u+c, u-c) \in \mathbb{K}(\Psi_{u+c}(u), \Psi_{u-c}(u)) \subset \mathbb{K}(\Psi(u)).$$

By induction we deduce that

$$\psi_0(u), \dots, \psi_n(u) \in \mathbb{K}(\Psi(2^{-N}u)),$$

for each $N \in \mathbb{N}$. Hence since $\Psi(2^{-N}u)$ is convergent on $U_n(2^N\epsilon)$, Ψ is also convergent on $U_n(2^N\epsilon)$. Thus each ψ_i is a meromorphic function and therefore by Remark 1.1 it is the quotient of two convergent power series, whose complex domain of convergence is $\mathbb{C}^n$. $\square$

Corollary 1.12. *Any $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT is algebraic over $\mathbb{K}(\psi)$ for some $\psi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT and whose coordinate functions are the quotient of two convergent power series whose complex domain of convergence is $\mathbb{C}^n$.*

Proof. Let $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admit an AAT. By Theorem 1.11, there exists $\psi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT whose coordinate functions are the quotient of two convergent whose complex domain of convergence is $\mathbb{C}^n$ and such that ψ is algebraic over $\mathbb{K}(\phi)$. Since the coordinate functions of ψ are algebraically independent, ϕ is algebraic over $\mathbb{K}(\psi)$. $\square$

We end this section with some basic properties of differentials that will be needed for the proof of Theorem 3.8. We introduce the following notation. Let $u := (u_1, \dots, u_n)$ be n variables, then for any $j \in \{1, \dots, n\}$ we denote $\partial_{u_j} : \mathcal{O}_{\mathbb{K},n} \to \mathcal{O}_{\mathbb{K},n}$ the formal derivative in the variable u_j. As ∂_{u_j} is a derivation of $\mathcal{O}_{\mathbb{K},n}$ it induces a derivation on $\mathcal{M}_{\mathbb{K},n}$. Given $\phi \in \mathcal{M}_{\mathbb{K},n}$ let $d\phi$ be the differential of ϕ, *i.e.*, $[\partial_{u_1}\phi, \dots, \partial_{u_n}\phi]$. We note that if ${}^{\mathrm{a}}\phi$ is the germ of an analytic function at 0 then ${}^{\mathrm{a}}(d\phi)$ is the gradient $\nabla\, {}^{\mathrm{a}}\phi$ of ${}^{\mathrm{a}}\phi$.

Lemma 1.13. *Let $\phi \in \mathcal{M}^m_{\mathbb{K},n}$ be such that $d\phi_1, \dots, d\phi_m$ are linearly independent over $\mathcal{M}_{\mathbb{K},n}$. Then*

(1) *$\phi_1, \dots, \phi_m$ are algebraically independent over $\mathbb{K}$.*
(2) *If $\psi \in \mathcal{M}^m_{\mathbb{K},n}$ and ϕ is algebraic over $\mathbb{K}(\psi)$, then $d\psi_1, \dots, d\psi_m$ are linearly independent over $\mathcal{M}_{\mathbb{K},n}$.*

PROOF. (1) Suppose that $\phi_1, \ldots, \phi_m$ are algebraically dependent over $\mathbb{K}$, then we may assume that ϕ_m is algebraic over $\mathbb{K}(\phi_1, \ldots, \phi_{m-1})$. If ϕ_m is constant then $d\phi_m = 0$, which is a contradiction, so we may assume that $\phi_m \notin \mathbb{K}$.

Let P be the minimal polynomial of ϕ_m over $\mathbb{K}(\phi_1, \ldots, \phi_{m-1})$. Note that $P(\phi_m)$ and $\frac{\partial P}{\partial X}(\phi_m)$ are elements of $\mathscr{M}_{\mathbb{K},n}$ and

$$dP(\phi_m) = \sum_{i=1}^{m-1} g_i \, d\phi_i + \frac{\partial P}{\partial X}(\phi_m) \, d\phi_m$$

for some $g_1, \ldots, g_{m-1} \in \mathscr{M}_{\mathbb{K},n}$. Since P is the minimal polynomial of ϕ_m, $P(\phi_m) = 0$. This implies that $dP(\phi_m)$ is the vector $[0, \ldots, 0]$ of $\mathscr{M}_{\mathbb{K},n}^n$ and there exist $h_1, \ldots, h_{m-1} \in \mathscr{M}_{\mathbb{K},n}$ such that

$$d\phi_m = \sum_{i=1}^{m-1} h_i \, d\phi_i,$$

which is a contradiction because $d\phi_1, \ldots, d\phi_m$ are linearly independent over $\mathscr{M}_{\mathbb{K},n}$.

(2) For every $i \in \{1, \ldots, m\}$ we have that ϕ_i is not constant because $d\phi_i \neq 0$. Since ϕ is algebraic over $\mathbb{K}(\psi)$, by the proof of (1) for each $i \in \{1, \ldots, m\}$ there exist $g_{i,1}, \ldots, g_{i,m} \in \mathscr{M}_{\mathbb{K},n}$ such that

$$d\phi_i = \sum_{j=1}^{m} g_{i,j} \, d\psi_j.$$

Therefore there exists a $m \times m$ matrix G with coefficients in $\mathscr{M}_{\mathbb{K},n}$ such that

$$\begin{bmatrix} d\phi_1 \\ \vdots \\ d\phi_m \end{bmatrix} = G \begin{bmatrix} d\psi_1 \\ \vdots \\ d\psi_m \end{bmatrix}.$$

Since $\phi_1, \ldots, \phi_m$ are algebraically independent over $\mathbb{K}$ by (1), we have that ψ is algebraic over $\mathbb{K}(\phi)$ and hence, by symmetry, there exists a $m \times m$ matrix H with coefficients in $\mathscr{M}_{\mathbb{K},n}$ such that

$$\begin{bmatrix} d\psi_1 \\ \vdots \\ d\psi_m \end{bmatrix} = H \begin{bmatrix} d\phi_1 \\ \vdots \\ d\phi_m \end{bmatrix}.$$

Hence $HG = GH = Id$, and so $d\psi_1, \ldots, d\psi_m$ are linearly independent over $\mathscr{M}_{\mathbb{K},n}$. $\square$

Chapter 2

Locally $\mathbb{K}$-Nash groups

The (real) locally Nash category was first introduced by M. Shiota in [**43**] and used in the context of groups in J.J Madden and C.M. Stanton [**26**] (see also Shiota's approach [**44**]) to study universal coverings of Nash groups and to classify one-dimensional Nash groups.

We will consider the analogous category in the complex case, and for that reason we will write a detailed presentation encompassing both the real and complex settings. For the rest of this memoir we will reserve the word Nash just for the real case and we will use $\mathbb{C}$-Nash for the complex case, or $\mathbb{K}$-Nash if we want to talk about both cases simultaneously. We point out that the complex version is based on the notion of $\mathbb{C}$-Nash, applied earlier in the solution of approximation problems in J. Adamus and S. Randriambololona [**3**], A. Tancredi and A. Tognoli [**46**] and others (see also J.M. Ruiz [**39**, Ch. 5 §5]). Generalizations of this concept for algebraically closed fields of characteristic 0 have also been considered by Y. Peterzil and S. Starchenko [**32, 33**] and M. Knebusch and R. Huber [**21, 24**].

Therefore, the main aim of this chapter is to develop the category of locally $\mathbb{K}$-Nash groups. In Section 1, we define $\mathbb{K}$-Nash map and we prove some of its basic properties. In Section 2, we define the category of locally $\mathbb{K}$-Nash maps. In Section 3, we define the category of locally $\mathbb{K}$-Nash groups, we study the atlas of such groups and we give sufficient and necessary conditions for two such groups to be isomorphic (Propositions 2.9 and 2.11). In Section 4, we will show that the category of locally $\mathbb{K}$-Nash groups is closed under universal coverings, see Proposition 2.13 (and also closed by quotients by normal discrete subgroups, see Remark 2.14). Finally, in Section 5, we will show that algebraic groups admit a natural locally $\mathbb{C}$-Nash group structure and we will study the behavior of locally $\mathbb{C}$-Nash maps between algebraic groups. In fact, as we already mentioned, E. Hrushovski and A. Pillay showed (see [**19**] and [**20**]) that locally Nash groups are precisely quotients of universal coverings of real algebraic groups by discrete subgroups. We will provide a proof of the analogous result for abelian locally $\mathbb{C}$-Nash groups in Chapter 3. There we will relate abelian locally $\mathbb{C}$-Nash groups with maps admitting an AAT, in order to use the results of Chapter 1. Throughout this chapter we will work with locally $\mathbb{K}$-Nash groups in general, with no commutativity assumption.

1. $\mathbb{K}$-Nash maps

All along this chapter $\mathbb{K}$ will be $\mathbb{C}$ or $\mathbb{R}$. Before defining $\mathbb{K}$-Nash maps, we need some notation.

When we refer to semialgebraic subsets of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we mean semialgebraic as subsets of $\mathbb{R}^{2n}$ and similarly for semialgebraic maps.

Let U, V and W be open subsets of $\mathbb{K}^p$ with $W \subset U \cap V$. Given maps $f(f_1 \dots, f_m) : U \to \mathbb{K}^m$ and $g : V \to \mathbb{K}^n$ we say that f is *algebraic* over $\mathbb{K}(g) := \mathbb{K}(g_1, \dots, g_n)$ on W if each of its components $f_1, \dots, f_m$ are algebraic over $\mathbb{K}(g)$ on W, that is, for each $i \in \{1, \dots, m\}$ there exists a polynomial $P_i \in \mathbb{K}[X_1, \dots, X_n, Y]$ of positive degree in Y such that

$$P_i(g_1(u), \dots, g_n(u), f_i(u)) = 0 \text{ for each } u \in W.$$

We will denote id the identity map, $\mathrm{id} : \mathbb{K}^n \to \mathbb{K}^n : u \mapsto u$ on $\mathbb{K}^n$.

In the real context, Nash maps are those that are both smooth and semialgebraic. Alternatively, Nash maps are those whose domain of definition is an open semialgebraic set and that are both analytic and algebraic over $\mathbb{R}(\mathrm{id})$ (see e.g. J. Bochnak, M. Coste, M.-F. Roy [**6**, Proposition 8.1.8]). This definition can be extended to the complex case as follows.

Definition. *Let U be an open subset of $\mathbb{K}^n$. We say that $f : U \to \mathbb{K}^m$ is a $\mathbb{K}$-Nash map if U is a semialgebraic and f is analytic (as a $\mathbb{K}$-function) and algebraic over $\mathbb{K}(\mathrm{id})$ on U.*

In the complex case ($\mathbb{K} = \mathbb{C}$), these maps have been studied for example by A. Tancredi and A. Tognoli in [**46**]. Since the word Nash is usually reserved for the real case, we use Nash (instead of $\mathbb{R}$-Nash), $\mathbb{C}$-Nash and $\mathbb{K}$-Nash for the real, complex and general case, respectively. This notation will allow us to give uniform proofs that work both for the real and the complex case.

We remark that although the real and complex case have analogous definitions and share similar properties, there are also some subtle differences, because the complex case is more rigid than the real one. For example, in the real case any semialgebraic function becomes a Nash function, when restricted to an adequate open dense subset of its domain of definition, see e.g. J.F. Fernando, J.M. Gamboa, J.M. Ruiz, [**13**, 2.4.1].

Fact 2.1 ([**13**, 2.4.1]). *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a semialgebraic map. Then, there exists an open dense subset $V \subset U$ such that $f : V \to \mathbb{R}^m$ is Nash.*

Proof. Fix n and $U \subset \mathbb{R}^n$. We say that $g : U \to \mathbb{R}$ has complexity $\leq d$ if there is a non-zero polynomial P in $n+1$ variables with coefficients in $\mathbb{R}$ of total degree $\leq d$, such that $P(x, g(x)) = 0$ for all $x \in U$. We denote by $S^k(U)$ the set of all semialgebraic functions from U to $\mathbb{R}$ such that all its partial derivatives up to order k exist and are continuous. We note that, by the proof of [**6**, Lemma 2.5.2.], for each $i \in \{1, \dots, m\}$, there exists a polynomial $P_i \in \mathbb{R}[X_1, \dots, X_{n+1}]$ such that $P_i(x, f_i(x)) = 0$, for every $x \in U$. Hence there exists $N \in \mathbb{N}$ such that all of $f_1, \ldots, f_m$ have complexity $\leq N$. By [**6**, Theorem 8.10.5] there exists $r = r(n, N)$ such that for every open semialgebraic subset V of $\mathbb{R}^n$, every function that belongs to $S^r(V)$ of complexity $\leq N$ is Nash. Take a C^r cell decomposition of the graph of f (see e.g. L. van den Dries [**47**, Ch.7 §3.3]). Consider the union of all cells of dimension n and let V be its projection over $\mathbb{R}^n$. Then the set V is an open dense subset of U and $f|_V$ is Nash. □

In contrast with the real case, the following well-known result holds only for the complex case.

FACT 2.2 ([**15**, Ch.III, B.13]). *Let U be an open subset of $\mathbb{C}^n$ and let $f : U \to \mathbb{C}^m$ be continuous and algebraic over $\mathbb{C}(\mathrm{id})$. Then f is analytic. In particular, if U is a semialgebraic set then f is a $\mathbb{C}$-Nash map.*

PROOF. We reproduce the proof in [**15**] for the sake of completeness. It is enough to prove the case $m = 1$. We will show that f is analytic at each point of U. Take $P \in \mathbb{C}[X_1, \ldots, X_n, Y]$, $P \neq 0$, such that $P(u, f(u)) \equiv 0$ on U. Without loss of generality we may assume that P is of the form $\prod_{i=1}^{d} P_i$, for some $d \in \mathbb{N}$, where each P_i is an irreducible polynomial in $\mathbb{C}[X_1, \ldots, X_n, Y]$ and $P_i \neq P_j$ for each $i \neq j$. Let $D \subset \mathbb{C}^n$ be the set of points where the discriminant of P with respect to Y vanishes. We note that D is an analytic subset of U. We also note that $D \neq U$, because P cannot have multiple roots since each P_i is irreducible and $P_i \neq P_j$ if $i \neq j$. So $U \setminus D$ is an open dense subset of U. Take $a \in U \setminus D$. Since $P(a, f(a)) = 0$ and $(\partial P/\partial Y)(a, f(a)) \neq 0$, by the Implicit Function Theorem, there is a unique function g, analytic on a neighborhood W of a, such that $P(u, g(u)) \equiv 0$ on W and $g(a) = f(a)$. But since $P(u, f(u)) \equiv 0$ on W, we get that $f \equiv g$ on W and hence f is analytic on W. This argument shows that f is analytic on $U \setminus D$. Since f is continuous in U, in particular f is locally bounded on U, so we can use the Riemann Removable Singularity Theorem (see e.g. [**15**, Ch.I, C.3]) to deduce that f is analytic on all U. □

The main aim of this section is to show that the conditions of being algebraic over $\mathbb{K}(\mathrm{id})$ and being semialgebraic are equivalent for analytic functions with a semialgebraic domain of definition.

PROPOSITION 2.3. *Let U be an open subset of $\mathbb{K}^n$. Then $f : U \to \mathbb{K}^m$ is both analytic and semialgebraic if and only if f is a $\mathbb{K}$-Nash map.*

PROOF. We first note that if f is semialgebraic then U is semialgebraic, so we may assume that U is semialgebraic. For $\mathbb{K} = \mathbb{R}$ this is [**6**, Proposition 8.1.8.]. So we prove the case $\mathbb{K} = \mathbb{C}$. We may assume that $m = 1$.

We begin with the left to right implication (a different proof can be found in [**3**, Proposition 4]). Assume first that U is an open connected semialgebraic neighborhood of the origin. Moreover, without loss of generality, U is invariant with respect to complex conjugation. As f is semialgebraic on U, then $\overline{f(\overline{z})}$ is also semialgebraic on U, because complex conjugation is a semialgebraic map. Consequently, the invariant analytic functions

$$g : U \to \mathbb{C} : z \mapsto \frac{f(z) + \overline{f(\overline{z})}}{2}$$

and

$$h : U \to \mathbb{C} : z \mapsto \frac{f(z) - \overline{f(\overline{z})}}{2i}$$

are semialgebraic and satisfy $f = g + ih$. Thus, $g|_{U\cap\mathbb{R}^n}$ and $h|_{U\cap\mathbb{R}^n}$ are analytic and semialgebraic functions, that is, Nash functions. Hence, there exist $G, H \in \mathbb{R}[X_1, \ldots, X_n, Y]\setminus\{0\}$ such that both $G(x, g(x))$ and $H(x, h(x))$ vanish identically on $U \cap \mathbb{R}^n$. Then, $G(z, g(z))$ and $H(z, h(z))$ also vanish identically on U by the identity principle, because U is connected. So $g(z)$ and $h(z)$ are algebraic over $\mathbb{C}(z)$ on U and hence $f(z) = g(z) + ih(z)$ is algebraic over $\mathbb{C}(z)$ on U.

For the general case, let U be an open semialgebraic subset of $\mathbb{C}^n$ and let $V_1, \ldots, V_d$ be the connected components of U, which are semialgebraic. Fix $i \in \{1, \ldots, d\}$ and $a \in V_i$ and let $V_i - a$ denote the set $\{z \in \mathbb{C}^n \mid z + a \in V_i\}$. By definition, $(V_i - a) \cap \overline{(V_i - a)}$ is an open connected semialgebraic neighborhood of 0. We also note that $f(z + a)$ is semialgebraic and analytic on $(V_i - a) \cap \overline{(V_i - a)}$. So we are in the hypothesis of the previous case and hence $f(z + a)$ is algebraic over $\mathbb{C}(z)$ on $(V_i - a) \cap \overline{(V_i - a)}$ and, by the identity principle, on $V_i - a$. This implies that $f(z)$ is algebraic over $\mathbb{C}(z - a)$ on V_i and, since $\mathbb{C}(z - a) \equiv \mathbb{C}(z)$, that $f(z)$ is algebraic over $\mathbb{C}(z)$ on V_i. So there exists a polynomial $P_i \in \mathbb{C}[X_1, \ldots, X_n, Y] \setminus \{0\}$ such that $P_i(z, f(z))$ vanishes identically in V_i. Repeating this process for each $i \in \{1, \ldots, d\}$ and letting $P := P_1 \cdot \ldots \cdot P_d$ we get that $P(z, f(z))$ vanishes identically on U. So $f(z)$ is algebraic over $\mathbb{C}(z)$ on U.

To prove the right to left implication, it is enough to show that if $f : U \to \mathbb{C}$ is an analytic function on a semialgebraic open subset U of $\mathbb{C}^n$ such that $P(x, f(x)) = 0$ on U for some $P \in \mathbb{C}[X_1, \ldots, X_n, X]$, $P \neq 0$, then f is a semialgebraic function. Firstly, we note that the addition and multiplication of complex numbers are semialgebraic maps, since both

$$+_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, y_1), (x_2, y_2) \mapsto (x_1 + x_2, y_1 + y_2)$$

and

$$\cdot_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, y_1), (x_2, y_2) \mapsto (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

are semialgebraic maps. This also shows that for any $Q \in \mathbb{C}[X_1, \ldots, X_p]$, the map that assigns to each $z \in \mathbb{C}^p$ its value $Q(z)$ is semialgebraic. In particular,

$$A := \{(x, y, u, v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid P(x + iy, u + iv) = 0\}$$

is a semialgebraic set. We note that the graph of f is a subset of A, so to show that f is semialgebraic we will find a partition of A into semialgebraic subsets compatible with the graph f. Now, take $N \in \mathbb{N}$ such that for each $z \in \mathbb{C}^n$ the number of roots of $P(z, X)$ is bounded by N. Fix $k \leq N$ and let

$$Z_k := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |A \cap ((x, y) \times \mathbb{R}^2)| = k\}.$$

Since A is a semialgebraic set, Z_k is also a semialgebraic set. Now, for each $1 \leq j \leq k$, let $s_j : Z_k \to \mathbb{C}$ be the semialgebraic map whose graph is

$$\{(z, r) \in (Z_k \times \mathbb{R}^2) \cap A \mid \exists r_1 \ldots \exists r_k \in \mathbb{R}^2, r_1 < \ldots < r_k,$$
$$(z, r_1) \ldots, (z, r_k) \in (Z_k \times \mathbb{R}^2) \cap A \text{ and } r = r_j\},$$

where $<$ denotes the lexicographic order of $\mathbb{R}^2$. Take a cell decomposition of $\mathbb{R}^{2n}$ compatible with the sets $Z_1, \dots, Z_N, U$ such that each s_j is continuous in each one of the cells. So take $C \subset U$ one of these cells and for each $j \in \{1, \dots, k\}$ let $C_j := \{x \in C \mid s_j(x) = f(x)\}$. Since, for each j, both f and s_j are continuous in C, each C_j is closed in C. Also, by definition of the s_j, the sets $C_1, \dots, C_k$ are disjoint and, as they are finitely many, they are open in C. Since C is connected and not empty, we deduce that there exists a unique $j \in \{1, \dots, k\}$ such that $C = C_j$. So $f \equiv s_j$ on C, which shows that the restriction of f to C is semialgebraic. This is clearly enough, since we have partitioned $\mathbb{R}^{2n}$ into finitely many semialgebraic sets $Z_1, \dots, Z_N$, and each one of these sets into finitely many cells like C. □

By Proposition 2.3, it is equivalent to consider maps that are semialgebraic or are algebraic over $\mathbb{K}(\mathrm{id})$, as long as we are in the analytic category. Throughout this memoir, we will think $\mathbb{K}$-Nash maps as analytic and algebraic defined on a semialgebraic set.

The following properties will be frequently used in our proofs. For the real case they are well known (see e.g. [**6**]). For completeness, we will provide a proof for the general case.

Lemma 2.4.

(1) *The restriction of a $\mathbb{K}$-Nash map to an open semialgebraic subset of the semialgebraic domain of definition is a $\mathbb{K}$-Nash map.*
(2) *Given $\mathbb{K}$-Nash maps $f : U \subset \mathbb{K}^m \to \mathbb{K}^n$ and $g : V \subset \mathbb{K}^n \to \mathbb{K}^p$ with $f(U) \subset V$, the composition $g \circ f$ is a $\mathbb{K}$-Nash map.*
(3) *Given a $\mathbb{K}$-Nash map $f : U \times V \subset \mathbb{K}^m \times \mathbb{K}^n \to \mathbb{K}^p$ and $a \in V$, the evaluation of $f(x, y)$ at $y = a$ is also a $\mathbb{K}$-Nash map, $f(x, a) : U \to \mathbb{K}^p$.*
(4) *Given a $\mathbb{K}$-Nash map $f : U \subset \mathbb{K}^n \to \mathbb{K}^n$, if f is an analytic diffeomorphism then f^{-1} is also a $\mathbb{K}$-Nash map.*

Proof. (1) It is trivial from the definition.

(2) Since $g(x)$ is algebraic over $\mathbb{K}(x)$ on V, we also have that $g(f(x))$ is algebraic over $\mathbb{K}(f(x))$ on U. Since $f(x)$ is algebraic over $\mathbb{K}(x)$ on U, we get that $g(f(x))$ is algebraic over $\mathbb{K}(x)$ on U.

(3) It is trivial noting that the projection of a semialgebraic set is a semialgebraic set (which is the Tarski-Seidenberg Theorem [**6**, Theorem 2.2.1]) and using Proposition 2.3. We provide however an alternative proof considering f as an analytic algebraic map. Suppose first that U and V are connected neighborhoods of the identity. Since $f(x, y)$ is algebraic over $\mathbb{K}(x, y)$ on $U \times V$, by Lemma 1.4 we get that $f(x, a)$ is algebraic over $\mathbb{K}(x)$ on a sufficiently small neighborhood of the identity and hence on U. Since $f(x, y)$ is analytic, $f(x, a)$ is also analytic so we are done with this case. The general case is proved as in the left to right implication of Proposition 2.3 with the appropiate modifications.

(4) It is enough to show that f^{-1} is algebraic over $\mathbb{K}(\mathrm{id})$. By hypothesis f is algebraic over $\mathbb{K}(\mathrm{id})$ on U. Hence id is algebraic over $\mathbb{K}(f^{-1})$ on $f(U)$. So, f^{-1} is algebraic over $\mathbb{K}(\mathrm{id})$ on $f(U)$, as required. □

2. Locally $\mathbb{K}$-Nash manifolds

In this section we collect the definitions and basic properties related to locally $\mathbb{K}$-Nash manifolds.

Let U be an open subset of $\mathbb{K}^m$. We say that $f : U \to V \subset \mathbb{K}^n$ is a $\mathbb{K}$-*Nash diffeomorphism* if f is an analytic diffeomorphism and both f and f^{-1} are $\mathbb{K}$-Nash maps (see also Lemma 2.4.(4)). Let M be a $\mathbb{K}$-analytic manifold. Two charts (U, ϕ) and (V, ψ) of an atlas for M are $\mathbb{K}$-*Nash compatible* if $\phi(U)$ and $\psi(V)$ are semialgebraic and either $U \cap V = \emptyset$ or

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a $\mathbb{K}$-Nash diffeomorphism. An atlas of M is a $\mathbb{K}$-*Nash atlas* if any two charts in the atlas are $\mathbb{K}$-Nash compatible. In particular, $\phi(U)$ is semialgebraic for any (U, ϕ) in the $\mathbb{K}$-Nash atlas. A $\mathbb{K}$-analytic manifold M together with a $\mathbb{K}$-Nash atlas is called a *locally* $\mathbb{K}$-*Nash manifold* (the word "locally" is added since the term $\mathbb{K}$-Nash manifold is usually reserved for those locally $\mathbb{K}$-Nash manifolds that admit a finite $\mathbb{K}$-Nash atlas).

Let M_1 and M_2 be locally $\mathbb{K}$-Nash manifolds equipped with $\mathbb{K}$-Nash atlases $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ respectively. A *locally* $\mathbb{K}$-*Nash map* $f : M_1 \to M_2$ is a (continuous) map such that for every $a \in M_1$ and every $j \in J$ such that $f(a) \in V_j$ there exists $i \in I$ and an open subset $U \subset U_i$ such that $a \in U$, $f(U) \subset V_j$ and

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U) \to \psi_j(V_j)$$

is a $\mathbb{K}$-Nash map. (For an equivalent definition see Proposition 2.5.) A locally $\mathbb{K}$-Nash map $f : M_1 \to M_2$ is a *locally* $\mathbb{K}$-*Nash diffeomorphism* if f is an analytic (global) diffeomorphism and both f and f^{-1} are locally $\mathbb{K}$-Nash maps.

Locally $\mathbb{K}$-Nash maps can be characterized as follows.

Proposition 2.5. *Let M_1 and M_2 be locally $\mathbb{K}$-Nash manifolds with $\mathbb{K}$-Nash atlases $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ respectively. The following are equivalent:*

(1) *$f : M_1 \to M_2$ is a locally $\mathbb{K}$-Nash map.*

(2) *For every $a \in M_1$ and for each $i \in I$ and $j \in J$ such that $a \in U_i$ and $f(a) \in V_j$ there exists an open subset U of U_i such that $a \in U$, $f(U) \subset V_j$, and*

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U) \to \psi_j(V_j)$$

is a $\mathbb{K}$-Nash map.

(3) *For every $a \in M_1$ there exist $i \in I$ and $j \in J$ such that $a \in U_i$ and $f(a) \in V_j$ and there exists an open subset U of U_i such that $a \in U$, $f(U) \subset V_j$, and*

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U) \to \psi_j(V_j)$$

is a $\mathbb{K}$-Nash map.

Proof. Since (2) implies (1) and (1) implies (3), it is enough to show that (3) implies (2). Fix $a \in M_1$ and let $i \in I$, $j \in J$ and $U \subset U_i$ be provided by (3). Fix $k \in I$ and $\ell \in J$ with $a \in U_k$ and $f(a) \in V_\ell$. Clearly, it suffices to show that there exists an open subset U' of U_k with $a \in U'$ such that

$$\psi_\ell \circ f \circ \phi_k^{-1} : \phi_k(U') \to \psi_\ell(V_\ell)$$

is $\mathbb{K}$-Nash. To prove the latter, firstly note that $\psi_j \circ f \circ \phi_i^{-1}$ is continuous and both $U_i \cap U_k \ni a$ and $V_j \cap V_\ell \ni f(a)$ are open, hence there exists an open subset U' of $U \cap U_k \subset U_i \cap U_k$ with $a \in U'$ such that

$$(\psi_j \circ f \circ \phi_i^{-1})(\phi_i(U')) \subset \psi_j(V_j \cap V_\ell).$$

Moreover, we can assume that $\phi_i(U')$ is semialgebraic (it suffices to take, instead of U', the preimage of an open ball centered in $\phi_i(a)$ and contained in the original $\phi_i(U')$). In particular, since the restriction of the $\mathbb{K}$-Nash map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U) \to \psi_j(V_j)$ to an open semialgebraic set is a $\mathbb{K}$-Nash map, the map

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U') \to \psi_j(V_j \cap V_\ell)$$

is still a $\mathbb{K}$-Nash map. On the other hand, both change of charts

$$\phi_i \circ \phi_k^{-1} : \phi_k(U') \to \phi_i(U')$$

and

$$\psi_\ell \circ \psi_j^{-1} : \psi_j(V_j \cap V_\ell) \to \psi_\ell(V_j \cap V_\ell)$$

are $\mathbb{K}$-Nash maps. Thus, the composition of the last three maps,

$$\psi_\ell \circ f \circ \phi_k^{-1} = (\psi_\ell \circ \psi_j^{-1}) \circ (\psi_j \circ f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_k^{-1}) : \phi_k(U') \to \psi_\ell(V_\ell)$$

is a $\mathbb{K}$-Nash map, as required. □

From Proposition 2.5.(2) it is clear that the composition of locally $\mathbb{K}$-Nash maps is a locally $\mathbb{K}$-Nash map. We also deduce the following.

Corollary 2.6. *Let M_1 and M_2 be locally $\mathbb{K}$-Nash manifolds. Then $f : M_1 \to M_2$ is a locally $\mathbb{K}$-Nash diffeomorphism if and only if f is both an analytic diffeomorphism and a locally $\mathbb{K}$-Nash map.*

Proof. We prove the right to left implication, the other one is trivial. Let $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ be the $\mathbb{K}$-Nash atlases of M_1 and M_2 respectively. It suffices to prove that $f^{-1} : M_2 \to M_1$ is a locally $\mathbb{K}$-Nash map. Fix $a \in M_2$ and $i \in I$ such that $f^{-1}(a) \in U_i$. We have to show that there exists $j \in J$ and an open subset $V \subset V_j$ such that $a \in V$, $f^{-1}(V) \subset U_i$, $\psi_j(V)$ is semialgebraic and

$$\phi_i \circ f^{-1} \circ \psi_j^{-1} : \psi_j(V) \to \phi_i(U_i)$$

is a $\mathbb{K}$-Nash map. Let $j \in J$ be such that $a \in V_j$. For these $f^{-1}(a) \in M_2$, i and j, we can apply Proposition 2.5.(2) because f is a locally $\mathbb{K}$-Nash map and get an open subset U of U_i such that $f^{-1}(a) \in U$, $f(U) \subset V_j$ and

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U) \to \psi_j(V_j)$$

is a $\mathbb{K}$-Nash map. Therefore, the given j and $V := f(U)$ satisfy the required conditions once we note that the inverse of a $\mathbb{K}$-Nash map, which is an analytic diffeomorphism, is also a $\mathbb{K}$-Nash map. □

3. Locally $\mathbb{K}$-Nash groups

We can define next the key concept of this memoir.

Definition. A *locally* $\mathbb{K}$*-Nash group* is a locally Nash manifold equipped with group operations (multiplication and inversion) which are given by locally $\mathbb{K}$-Nash maps. A *homomorphism of locally* $\mathbb{K}$*-Nash groups* is a locally $\mathbb{K}$-Nash map that is also a homomorphism of groups. An *isomorphism of locally* $\mathbb{K}$*-Nash groups* is a locally $\mathbb{K}$-Nash diffeomorphism that is also an isomorphism of groups.

Next, we show how to describe the locally $\mathbb{K}$-Nash structure of a locally $\mathbb{K}$-Nash group via a chart of the identity. Let $(G,\cdot)$ equipped with an analytic atlas $\mathcal{A}$ be a $\mathbb{K}$-analytic group – thus a Lie group – and let (U,ϕ) be a chart in $\mathcal{A}$ of the identity. From the theory of analytic groups we recall that

$$\mathcal{A}_{(U,\phi)} := \{(gU,\phi_g) \mid \phi_g : gU \to \mathbb{K}^n : u \mapsto \phi(g^{-1}u)\}_{g\in G}$$

is also an analytic atlas for $(G,\cdot)$. We will keep the notation $\mathcal{A}_{(U,\phi)}$ for this canonical atlas. $\mathcal{A}_{(U,\phi)}$ might not be a $\mathbb{K}$-Nash atlas for $(G,\cdot)$, but if it is so, then the locally $\mathbb{K}$-Nash group $(G,\cdot)$ equipped with $\mathcal{A}_{(U,\phi)}$ will be denoted $(G,\cdot,\phi|_U)$. (In the abelian case, locally $\mathbb{K}$-Nash group structures on $(\mathbb{K}^n,+)$ will be denoted by $(\mathbb{K}^n,+,f)$, where $f:\mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a meromorphic map admitting an AAT, see Remark 3.7)

Fact 2.7 ([**26**, Lemma 1]). *Let $(G,\cdot)$ be a $\mathbb{K}$-analytic group with atlas $\mathcal{A}$. Let $(U,\phi)\in\mathcal{A}$ be a chart of the identity such that:*

(i) *there exists an open neighborhood of the identity $U'\subset U$ such that*

$$\phi\circ\cdot\circ(\phi^{-1},\phi^{-1}) : \phi(U')\times\phi(U') \to \phi(U) : (x,y) \mapsto \phi(\phi^{-1}(x)\cdot\phi^{-1}(y))$$

is a $\mathbb{K}$-Nash map, and

(ii) *for each $g\in G$ there exists an open neighborhood of the identity $U_g\subset U$ such that*

$$\phi\circ -^g\circ\phi^{-1} : \phi(U_g)\to\phi(U) : x\mapsto \phi(g^{-1}\phi^{-1}(x)g)$$

is a $\mathbb{K}$-Nash map.

Then there exists a neighborhood of the identity $V\subset U$ such that $\mathcal{A}_{(V,\phi)} = \{(gV,\phi_g)\}_{g\in G}$ is a $\mathbb{K}$-Nash atlas for $(G,\cdot)$ and hence $(G,\cdot,\phi|_V)$ is a locally $\mathbb{K}$-Nash group.

We note that if $(G,\cdot)$ is an abelian group then (ii) of Fact 2.7 is trivially satisfied. So, in that case, the proposition says that each chart of the identity satisfying (i) induces a locally $\mathbb{K}$-Nash group structure on $(G,\cdot)$. We anticipate by means of Lemma 3.6 in Chapter 3 that a chart of the identity $(U,(\phi_1,\dots,\phi_n))$ of $(\mathbb{K}^n,+)$ with its standard analytic structure satisfies (i) if and only if it admits an algebraic addition theorem, *i.e.* if for some open neighborhood of the identity $U'\subset U$ and for each $i\in\{1,\dots,n\}$ there exists a $P_i\in\mathbb{K}[X_1,\dots,X_{2n+1}]$, $P_i\neq 0$, such that

$$P_i(\phi_1(u),\dots,\phi_n(u),\phi_1(v),\dots,\phi_n(v),\phi_i(u+v)) \equiv 0 \text{ on } U'\times U'.$$

Proof of Fact 2.7. First, given a chart of the identity $(U,\phi)\in\mathcal{A}$ satisfying (i) and (ii), we will find an open neighborhood of the identity

$V \subset U$ such that G equipped with $\mathcal{A}_{(V,\phi)} := \{(gV, \phi_g)\}_{g\in G}$ where

$$\phi_g : gV \to \mathbb{K}^n : u \mapsto \phi_g(u) = \phi(g^{-1}u)$$

is a locally $\mathbb{K}$-Nash manifold (for this, only (i) is needed). Then, we will check that $\cdot : G \times G \to G$ is a locally $\mathbb{K}$-Nash map when G is equipped with $\mathcal{A}_{(V,\phi)}$. Finally, we will show that $^{-1} : G \to G$ is a locally $\mathbb{K}$-Nash map when G is equipped with the atlas $\mathcal{A}_{(V,\phi)}$. This will complete the proof.

Since the map of (i) is continuous, there exists an open neighborhood of the identity $V \subset U'$ such that $V \cdot V \subset U'$ and $V = V^{-1}$. Moreover, we can assume that $\phi(V)$ is semialgebraic (it suffices to take the preimage of an open ball centered in $\phi(e)$, where e is the identity of the group, and contained in the original $\phi(U)$). We show that $\mathcal{A}_{(V,\phi)}$, as defined above, is a $\mathbb{K}$-Nash atlas for G. We note that for each $g \in G$

$$(\phi_g)^{-1} : \phi(V) \to gV : x \mapsto g\phi^{-1}(x).$$

So we have to check that if $g, h \in G$ are given such that $gV \cap hV \neq \emptyset$ then

$$\phi_h \circ (\phi_g)^{-1} : \phi(V \cap g^{-1}hV) \to \phi(V \cap h^{-1}gV) : x \mapsto \phi(h^{-1}g\phi^{-1}(x))$$

is a $\mathbb{K}$-Nash diffeomorphism. Since $V \cdot V \subset U'$ and $V = V^{-1}$, we have that $h^{-1}g \in U'$. Thus, we can evaluate the map of (i) at $(\phi(h^{-1}g), x)$ to deduce that

$$\phi_h \circ (\phi_g)^{-1} : \phi(U') \to \phi(U) : x \mapsto \phi(h^{-1}g\phi^{-1}(x))$$

is a $\mathbb{K}$-Nash map. Since $\phi(h^{-1}gV)$ is the image of $\phi(V)$ under $\phi_h \circ (\phi_g)^{-1}$ and $\phi(V)$ is semialgebraic, $\phi(h^{-1}gV)$ is also semialgebraic. We note that $\phi(V \cap h^{-1}gV)$ is equal to $\phi(V) \cap \phi(h^{-1}gV)$ and, hence, semialgebraic. Thus, the map

$$\phi_h \circ (\phi_g)^{-1} : \phi(V \cap g^{-1}hV) \to \phi(V \cap h^{-1}gV) : x \mapsto \phi(h^{-1}g\phi^{-1}(x))$$

is a $\mathbb{K}$-Nash map. By symmetry, the same argument shows that $\phi_g \circ (\phi_h)^{-1}$ is also a $\mathbb{K}$-Nash map. Since $\mathcal{A}_{(V,\phi)}$ is an analytic atlas for G, we have that $\phi_h \circ (\phi_g)^{-1}$ is an analytic diffeomorphism and, so a $\mathbb{K}$-Nash diffeomorphism. Therefore, G equipped with $\mathcal{A}_{(V,\phi)}$ is a locally $\mathbb{K}$-Nash manifold.

Now, we check that $\cdot : G \times G \to G$ is a locally $\mathbb{K}$-Nash map when G is equipped with $\mathcal{A}_{(V,\phi)}$. By Proposition 2.5.(3), it is enough to check that for each $g, h \in G$ there exist open neighborhoods of the identity $V_1, V_2 \subset V$ such that

$$\phi_{gh} \circ \cdot \circ ((\phi_g)^{-1}, (\phi_h)^{-1}) : \phi(V_1) \times \phi(V_2) \to \phi(V) : (x, y) \mapsto \phi(h^{-1}\phi^{-1}(x)h\phi^{-1}(y))$$

is a $\mathbb{K}$-Nash map. Reasoning as in the first part of the proof, and since the maps of (i) and (ii) for h are $\mathbb{K}$-Nash, there exist open neighborhoods of the identity $V_1', V_2 \subset V$ and $V_1 \subset V_1' \cap U_h$ such that both

$$\phi \circ \cdot \circ (\phi^{-1}, \phi^{-1}) : \phi(V_1') \times \phi(V_2) \to \phi(V) : (x, y) \mapsto \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

and

$$\phi \circ -^h \circ \phi^{-1} : \phi(V_1) \to \phi(V_1') : x \mapsto \phi(h^{-1}\phi^{-1}(x)h)$$

are $\mathbb{K}$-Nash maps. An adequate composition – which is also $\mathbb{K}$-Nash – of the latter gives the required map.

Next, we show that the map

$$\phi \circ {}^{-1} \circ \phi^{-1} : \phi(V) \to \phi(V) : x \mapsto \phi((\phi^{-1}(x))^{-1}) \tag{2.1}$$

is $\mathbb{K}$-Nash. Since the map of (2.1) is analytic, because $\mathcal{A}$ is an analytic atlas for $(G,\cdot)$, it is enough to check that it is algebraic over $\mathbb{K}(\mathrm{id})$. In the previous paragraph, we have shown that there exist open neighborhoods of the identity $V_1', V_2 \subset V$ such that

$$\phi \circ \cdot \circ (\phi^{-1}, \phi^{-1}) : \phi(V_1') \times \phi(V_2) \to \phi(V) : (x,y) \mapsto \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

is a $\mathbb{K}$-Nash map. In particular, this map is algebraic over $\mathbb{K}(\mathrm{id})$ on $\phi(V_1') \times \phi(V_2)$ and, hence, $\phi(x \cdot y)$ is algebraic over $\mathbb{K}(\phi(x), \phi(y))$ on $V_1' \times V_2$. Taking into account transcendence degrees, $\phi(y)$ is algebraic over $\mathbb{K}(\phi(x), \phi(x \cdot y))$ on $V_1' \times V_2$. So we may substitute y by x^{-1} to get that $\phi(x^{-1})$ is algebraic over $\mathbb{K}(\phi(x))$ on a sufficiently small neighborhood of the identity. In particular $\phi(\phi^{-1}(x)^{-1})$ is algebraic over $\mathbb{K}(x)$ as required.

Now we check that ${}^{-1} : G \to G$ is a locally $\mathbb{K}$-Nash map when G is equipped with $\mathcal{A}_{(V,\phi)}$. By Proposition 2.5.(3) it is enough to prove that for each $g \in G$ there exists an open neighborhood of the identity $V_1 \subset V$ such that

$$\phi_{g^{-1}} \circ {}^{-1} \circ (\phi_g)^{-1} : \phi(V_1) \to \phi(V) : x \mapsto \phi(g(\phi^{-1}(x))^{-1} g^{-1})$$

is a $\mathbb{K}$-Nash map. Reasoning once more as in the first part of the proof, and since the map of property (ii) for g^{-1} is $\mathbb{K}$-Nash, there exists an open neighborhood of the identity $V_1 \subset V$ such that

$$\phi \circ -^{g} \circ \phi^{-1} : \phi(V_1) \to \phi(V) : x \mapsto \phi(g\phi^{-1}(x)g^{-1})$$

is a $\mathbb{K}$-Nash map. Composing the latter map with the map in (2.1), we obtain a $\mathbb{K}$-Nash map, which is the map required to be $\mathbb{K}$-Nash. This completes the proof. $\square$

PROPOSITION 2.8. *Let $(G,\cdot)$ be a locally $\mathbb{K}$-Nash group equipped with a $\mathbb{K}$-Nash atlas $\mathcal{A}$. Then, for every chart of the identity $(U,\phi) \in \mathcal{A}$, there exists an open subset V of U such that $(G,\cdot)$ equipped with $\mathcal{A}$ is isomorphic to $(G,\cdot,\phi|_V)$.*

PROOF. Firstly, we will check that (U,ϕ) satisfies (i) and (ii) of Fact 2.7. Then, by Fact 2.7, there exists $V \subset U$ such that $\mathcal{A}_{(V,\phi)}$ is a $\mathbb{K}$-Nash atlas for $(G,\cdot)$. Finally, we will show that the identity map from G equipped with $\mathcal{A}$ to G equipped with $\mathcal{A}_{(V,\phi)}$ is a locally $\mathbb{K}$-Nash diffeomorphism and, hence, an isomorphism of locally $\mathbb{K}$-Nash groups.

Let $(U,\phi) \in \mathcal{A}$ be a chart of the identity. Since $\cdot : G \times G \to G$ is a locally $\mathbb{K}$-Nash map when G is equipped with $\mathcal{A}$, by Proposition 2.5.(2) we deduce the following facts:

(1) There exists an open neighborhood of the identity $U' \subset U$ such that

$$\phi \circ \cdot \circ (\phi^{-1}, \phi^{-1}) : \phi(U') \times \phi(U') \to \phi(U) : (x, y) \mapsto \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

is a $\mathbb{K}$-Nash map. So (U, ϕ) satisfies (i) of Fact 2.7.

(2) Fix $g \in G$ and $(W_1, \psi_1), (W_2, \psi_2) \in \mathcal{A}$ coordinate neighborhoods of g and g^{-1} respectively. Then there exist open neighborhoods $W_1' \subset W_1$ and $W_2' \subset W_2$ of g and g^{-1} respectively such that

$$\begin{array}{rcccl} \phi \circ \cdot \circ (\psi_2^{-1}, \psi_1^{-1}) & : & \psi_2(W_2') \times \psi_1(W_1') & \to & \phi(U) \\ & & (z, x) & \mapsto & \phi(\psi_2^{-1}(z) \cdot \psi_1^{-1}(x)) \end{array}$$

is a $\mathbb{K}$-Nash map. Similarly, there exist open neighborhoods $U_g \subset U$ and $W_1'' \subset W_1'$ of the identity and g respectively such that

$$\begin{array}{rcccl} \psi_1 \circ \cdot \circ (\phi^{-1}, \psi_1^{-1}) & : & \phi(U_g) \times \psi_1(W_1'') & \to & \psi_1(W_1') \\ & & (x, y) & \mapsto & \psi_1(\phi^{-1}(x) \cdot \psi_1^{-1}(y)) \end{array}$$

is a $\mathbb{K}$-Nash map. If we evaluate the first map at $z = \psi_2(g^{-1})$ and the second at $y = \psi_1(g)$, we again obtain $\mathbb{K}$-Nash maps. Then, composing both maps, we deduce that (U, ϕ) satisfies (ii) for g of Fact 2.7.

Hence, (U, ϕ) is under the hypothesis of Fact 2.7 and, therefore, there exists an open neighborhood of the identity $V \subset U$ such that $\mathcal{A}_{(V,\phi)}$ is a $\mathbb{K}$-Nash atlas for $(G, \cdot)$.

Now, we check that the identity map from G equipped with $\mathcal{A}$ to G equipped with $\mathcal{A}_{(V,\phi)}$ is a locally $\mathbb{K}$-Nash diffeomorphism. Since the identity map is an analytic diffeomorphism between the two analytic groups, by Corollary 2.6 it is enough to show that it is a locally $\mathbb{K}$-Nash map. By definition it suffices to show that for each $g, h \in G$ with $g \in hV$ there exist $(W_1, \psi_1) \in \mathcal{A}$ with $g \in W_1$ and an open neighborhood $W_1' \subset W_1 \cap hV$ of g such that $(\psi_1(W_1')$ is semialgebraic and)

$$\phi_h \circ \psi_1^{-1} : \psi_1(W_1') \to \phi(V) : x \mapsto \phi(h^{-1}\psi_1^{-1}(x))$$

is a $\mathbb{K}$-Nash map. Let g and h be fixed with $g \in hV$. Let $(W_2, \psi_2) \in \mathcal{A}$ be a coordinate neighborhood of h^{-1}. Assume G is equipped with $\mathcal{A}$. Since $h^{-1}g \in V$ and $\cdot : G \times G \to G$ is a locally $\mathbb{K}$-Nash map, there exists a coordinate neighborhood $(W_1, \psi_1) \in \mathcal{A}$ of g, and open neighborhoods $W_2' \subset W_2$ and $W_1' \subset W_1$ of h^{-1} and g respectively such that $W_2' \cdot W_1' \subset V$ and

$$\begin{array}{rcccl} \phi \circ \cdot \circ (\psi_2^{-1}, \psi_1^{-1}) & : & (\psi_2(W_2'), \psi_1(W_1')) & \to & \phi(V) \\ & & (x, y) & \mapsto & \phi(\psi_2^{-1}(x) \cdot \psi_1^{-1}(y)) \end{array}$$

is a $\mathbb{K}$-Nash map. We evaluate the map above at $x = \psi_2(h^{-1})$ to deduce that

$$\phi_h \circ \psi_1^{-1} : \psi_1(W_1') \to \phi(V) : x \mapsto \phi(h^{-1}\psi_1^{-1}(x))$$

is a $\mathbb{K}$-Nash map, as required. $\square$

The next proposition will provide a sufficient condition for a pure homomorphism of locally $\mathbb{K}$-Nash groups to be a locally $\mathbb{K}$-Nash homomorphism.

PROPOSITION 2.9. *Let G and H be locally $\mathbb{K}$-Nash groups and let $\alpha : G \to H$ be a group homomorphism. Suppose there are charts (U, ϕ) and*

(V,ψ) of G and H respectively and an open subset $U' \subset U$ such that $\alpha(U') \subset V$ and $\psi \circ \alpha \circ \phi^{-1} : \phi(U') \to \psi(V)$ is a $\mathbb{K}$-Nash map. Then α is a locally $\mathbb{K}$-Nash homomorphism.

PROOF. Firstly, we prove that α is a locally $\mathbb{K}$-Nash map, provided that $U' \subset U$ and V are neighborhoods of the identity of G and H respectively. By Proposition 2.8, we can assume that the locally $\mathbb{K}$-Nash groups G and H equipped with $\mathcal{A}_{(U,\phi)}$ and $\mathcal{A}_{(V,\psi)}$ are locally $\mathbb{K}$-Nash isomorphic to the original structures. Let $g \in G$. We have that (gU, ϕ_g) and $(\alpha(g)V, \psi_{\alpha(g)})$ are charts of G and H respectively with $g \in gU' \subset gU$ and $\alpha(g) \in \alpha(g)V$. By Proposition 2.5.(3), it would be enough to show that the map

$$\psi_{\alpha(g)} \circ \alpha \circ \phi_g^{-1} : \phi(U') \to \psi(V)$$

is $\mathbb{K}$-Nash. The latter is true since $\psi \circ \alpha \circ \phi^{-1}$ is a $\mathbb{K}$-Nash map and

$$\begin{aligned}(\psi_{\alpha(g)} \circ \alpha \circ \phi_g^{-1})(x) &= (\psi_{\alpha(g)} \circ \alpha)(g\phi^{-1}(x)) \\ &= \psi_{\alpha(g)}(\alpha(g)\alpha(\phi^{-1}(x))) \\ &= \psi(\alpha(g)^{-1}\alpha(g)\alpha(\phi^{-1}(x))) \\ &= \psi(\alpha(\phi^{-1}(x))).\end{aligned}$$

It remains to prove that we can assume that the relevant open sets can be taken neighborhoods of the identity. Fix $g \in U'$. Since the group operation of G is a locally $\mathbb{K}$-Nash map, there exist a chart (U_0, ϕ_0) of G and an open neighborhood of the identity $U_0' \subset U_0$ such that $gU_0' \subset U'$ and the map

$$L_g : \phi_0(U_0') \to \phi(U') : x \mapsto \phi(g\phi_0^{-1}(x))$$

is a $\mathbb{K}$-Nash map. Similarly, there exist a chart (V_0, ψ_0) of the identity of H and an open subset $V' \ni \alpha(g)$ of V such that $\alpha(g)^{-1}V' \subset V_0$ and

$$L_{\alpha(g)^{-1}} : \psi(V') \to \psi_0(V_0) : x \mapsto \psi_0(\alpha(g)^{-1}\psi^{-1}(x))$$

is a $\mathbb{K}$-Nash map. By continuity and since $(\psi\circ\alpha\circ\phi^{-1}\circ L_g)(\phi_0(e)) = \psi(\alpha(g))$, we can take U_0' small enough so that

$$(\psi \circ \alpha \circ \phi^{-1} \circ L_g)(\phi_0(U_0')) \subset \psi(V').$$

In particular, the composition

$$\begin{array}{rccl} L_{\alpha(g)^{-1}} \circ \psi \circ \alpha \circ \phi^{-1} \circ L_g & : & \phi_0(U_0') & \to & \psi_0(V_0) \\ & & x & \mapsto & \psi_0(\alpha(\phi_0^{-1}(x))) \end{array}$$

is a $\mathbb{K}$-Nash map, as required. □

REMARK 2.10. If $\mathbb{K} = \mathbb{R}$ in Proposition 2.9 then the result holds in case that $\psi \circ \alpha \circ \phi^{-1} : \phi(U') \to \psi(V)$ is just a semialgebraic map. Indeed, by Fact 2.1 and restricting U' if necessary, we can assume that the map $\psi \circ \alpha \circ \phi^{-1} : \phi(U') \to \psi(V)$ is Nash.

Next, we will characterize those analytic isomorphisms which are isomorphisms of locally $\mathbb{K}$-Nash groups. We recall that if an analytic map is an isomorphism of groups then its inverse is also an analytic map, see e.g. V.S. Varadarajan [**48**, Corollary 2.6.2]

Proposition 2.11. *Let G and H be locally Nash groups equipped with atlases $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, a continuous isomorphism $\alpha : G \to H$ is an isomorphism of locally $\mathbb{K}$-Nash groups if and only if there exist for each pair of charts of the identity $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ an open neighborhood of the identity $W \subset U \cap \alpha^{-1}(V)$ such that $\psi \circ \alpha$ is algebraic over $\mathbb{K}(\phi)$ on W.*

Proof. We begin with the left to right implication. Fix a pair of charts of the identity $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. Since α is a locally $\mathbb{K}$-Nash map, by Proposition 2.5.(2) there exists an open neighborhood of the identity $W \subset U \cap \alpha^{-1}(V)$ such that

$$\psi \circ \alpha \circ \phi^{-1} : \phi(W) \to \psi(V) : x \mapsto \psi(\alpha(\phi^{-1}(x)))$$

is a $\mathbb{K}$-Nash map. So $\psi \circ \alpha \circ \phi^{-1}$ is algebraic over $\mathbb{K}(\mathrm{id})$ on $\phi(W)$ and, hence, $\psi \circ \alpha$ is algebraic over $\mathbb{K}(\phi)$ on W, as required.

Next, we show the right to left implication for the case $\mathbb{K} = \mathbb{R}$. Fix $i \in \{1, \ldots, n\}$. By hypothesis $\psi_i \circ \alpha$ is algebraic over $\mathbb{R}(\phi)$ on W and, therefore, since ϕ is a diffeomorphism, $\psi_i \circ \alpha \circ \phi^{-1}$ is algebraic over $\mathbb{R}(\mathrm{id})$ on $\phi(W)$. Hence, there exists a polynomial $P \in \mathbb{R}[x][Y]$ such that $P(x, (\psi_i \circ \alpha \circ \phi^{-1})(x)) = 0$ for all $x \in \phi(W)$. Without loss of generality, we can assume that $\phi(W)$ is semialgebraic. Then, by the proof of [**6**, Proposition 8.1.8] and since $\psi_i \circ \alpha \circ \phi^{-1}$ is continuous, we obtain that each coordinate function $\psi_i \circ \alpha \circ \phi^{-1}$ is a semialgebraic function on $\phi(W)$. By Remark 2.10, we deduce that α is a locally Nash map. Moreover, we also have that the inverse of the above map,

$$\phi \circ \alpha^{-1} \circ \psi^{-1} : \psi(\alpha(W)) \to \phi(W) \subset \phi(U),$$

is semialgebraic and, therefore, again by Remark 2.10, we deduce that α^{-1} is a locally Nash map. Thus, α is a locally Nash isomorphism.

Finally, we prove the right to left implication of the case $\mathbb{K} = \mathbb{C}$. We can assume that $\phi(W)$ is connected and semialgebraic. Since $\psi \circ \alpha \circ \phi^{-1}|_{\phi(W)}$ is both continuous and algebraic over $\mathbb{C}(\mathrm{id})$, it follows from Fact 2.2 and Proposition 2.9 that α is a locally $\mathbb{C}$-Nash homomorphism. Moreover, by invariance of domain, we have that $V' := \alpha(W)$ is an open subset of V and $\psi \circ \alpha \circ \phi^{-1}|_{\phi(W)}$ an homeomorphism. Thus, we have that

$$\phi \circ \alpha^{-1} \circ \psi^{-1}|_{\psi(V')}$$

is continuous and also algebraic over $\mathbb{C}(\mathrm{id})$. Then, again by Fact 2.2 and Proposition 2.9, we conclude that α^{-1} is a locally $\mathbb{C}$-Nash isomorphism, as required. $\square$

The following is an immediate consequence of Proposition 2.11.

Corollary 2.12. *Let $(G, \cdot)$ equipped with a $\mathbb{K}$-Nash atlas $\mathcal{A}$ be a locally $\mathbb{K}$-Nash group. Let (U, ϕ) and (V, ψ) be charts of the identity of $\mathcal{A}$. If $(G, \cdot, \phi|_U)$ and $(G, \cdot, \psi|_V)$ are locally $\mathbb{K}$-Nash groups then they are locally $\mathbb{K}$-Nash isomorphic.*

Proof. Since (U, ϕ) and (V, ψ) are charts that are $\mathbb{K}$-Nash compatible,

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) : x \mapsto \psi \circ \phi^{-1}(x)$$

is a $\mathbb{K}$-Nash map. So ψ is algebraic over $\mathbb{K}(\phi)$ on $U \cap V$. Now, we can apply Proposition 2.11 taking α as the identity map. □

As a special case of Corollary 2.12, if $(G, \cdot, \phi|_U)$ and $(G, \cdot, \phi|_V)$ are locally $\mathbb{K}$-Nash groups, for some neighborhoods of the identity U and V, then they are isomorphic.

4. Universal coverings of locally $\mathbb{K}$-Nash groups

In this section we will show that the category of locally $\mathbb{K}$-Nash groups is closed under universal coverings and quotients by normal discrete subgroups. We first consider the universal coverings.

PROPOSITION 2.13. *Let G be a locally $\mathbb{K}$-Nash group. Let $(\widetilde{G}, \pi)$ be its (analytic) universal cover. Then, $\widetilde{G}$ can be equipped with the structure of a locally $\mathbb{K}$-Nash group in such a way that $\pi : \widetilde{G} \to G$ is a locally $\mathbb{K}$-Nash homomorphism. This structure is unique up to isomorphism. Any analytic section of the covering map is also a locally $\mathbb{K}$-Nash map.*

PROOF. By Proposition 2.8, we may assume that the $\mathbb{K}$-Nash atlas of G is of the form $\mathcal{A}_{(U,\phi)}$ for some open neighborhood of the identity $U \subset G$, where

$$\mathcal{A}_{(U,\phi)} := \{(gU, \phi_g) \mid \phi_g : gU \to \mathbb{K}^n : u \mapsto \phi(g^{-1}u)\}_{g \in G} .$$

We know that $\widetilde{G}$ is an analytic group and π an analytic homomorphism (see e.g. [**48**, Corollary 2.6.2]). Then, by [**48**, Theorem 2.9.5], there exist an open neighborhood of the identity $U' \subset U$ and an analytic section $s : U' \to \widetilde{G}$ such that $s(U')$ is a neighborhood of the identity of $\widetilde{G}$ and $\pi \circ s = Id$. Thus, $(s(U'), \phi \circ \pi)$ is a chart of $\widetilde{G}$. We note that since (U, ϕ) satisfies properties (i) and (ii) of Fact 2.7, $(s(U'), \phi \circ \pi)$ also does. So, by Fact 2.7, there exists an open neighborhood of the identity $V \subset s(U')$ such that $\widetilde{G}$ equipped with $\mathcal{A}_{(V,\phi\circ\pi)}$ is a locally $\mathbb{K}$-Nash group, where

$$\mathcal{A}_{(V,\phi\circ\pi)} := \{(gV, \phi_{\pi(g)} \circ \pi\}_{g \in \widetilde{G}} .$$

To check that the covering map π is a locally $\mathbb{K}$-Nash homomorphism, it is enough to apply Proposition 2.9 to π noting that

$$\phi \circ \pi \circ (\phi \circ \pi)^{-1} : \phi \circ \pi(V) \to \phi(U)$$

is the identity map and, hence, a $\mathbb{K}$-Nash map.

Now, we check that the structure is unique up to locally $\mathbb{K}$-Nash isomorphism. Fix two locally $\mathbb{K}$-Nash atlas $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\pi : \widetilde{G} \to G$ is a locally $\mathbb{K}$-Nash homomorphism when $\widetilde{G}$ is equipped with either $\mathcal{A}_1$ and $\mathcal{A}_2$. Since the analytic structure is unique up to analytic isomorphism, it is enough to check that $(\widetilde{G}, \mathcal{A}_1)$ and $(\widetilde{G}, \mathcal{A}_2)$ are locally $\mathbb{K}$-Nash isomorphic. Since π is a locally $\mathbb{K}$-Nash map, given a chart of the identity (φ, W) of G there exist charts of the identity $(\psi_1, V_1) \in \mathcal{A}_1$ and $(\psi_2, V_2) \in \mathcal{A}_2$ such that both $\varphi \circ \pi \circ \psi_1^{-1} : \psi_1(V_1) \to \mathbb{K}^n$ and $\varphi \circ \pi \circ \psi_2^{-1} : \psi_2(V_2) \to \mathbb{K}^n$ are $\mathbb{K}$-Nash diffeomorphisms. So $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ is a $\mathbb{K}$-Nash diffeomorphism. This means that ψ_2 is algebraic over $\mathbb{K}(\psi_1)$ on some open neighborhood of the identity and, hence, by Proposition 2.11, the identity map from $(\widetilde{G}, \mathcal{A}_1)$ to $(\widetilde{G}, \mathcal{A}_2)$ is a locally $\mathbb{K}$-Nash isomorphism.

Next, we fix an open subset $W \subset G$ and an analytic section $s_W : W \to \widetilde{G}$ and we want to check that s_W is a locally $\mathbb{K}$-Nash map. (We note that since W is an open subset of G, it has itself the structure of a locally $\mathbb{K}$-Nash manifold.) By Proposition 2.5.(3), it is enough to check that for each $a \in W$ there exist $g \in G$, $\tilde{g} \in \widetilde{G}$ and $U' \subset U$ such that the restriction of $(\phi_{\pi(\tilde{g})} \circ \pi) \circ s_W \circ \phi_g^{-1}$ to $\phi(U')$ is a $\mathbb{K}$-Nash map. Taking $g := a$, $\tilde{g} := s_W(a)$ and U' such that $a\phi(U')$ is a semialgebraic open subset of W, we get that the previous map is the identity map on $\phi(U')$ and, hence, a $\mathbb{K}$-Nash map. $\square$

We finish the section considering the quotients of locally $\mathbb{K}$-Nash groups by normal discrete subgroups.

REMARK 2.14. *Let $(G, \phi|_U)$ be a locally $\mathbb{K}$-Nash group and let Λ a normal discrete subgroup of G. Then, $(G/\Lambda, \phi|_U)$ is a locally $\mathbb{K}$-Nash group and the projection map is a locally $\mathbb{K}$-Nash map.*

PROOF. We can provide to G/Λ a structure of locally $\mathbb{K}$-Nash group as follows. Let $\pi : G \to G/\Lambda$ be the canonical projection of G onto G/Λ. Take (U, ϕ) a chart of G such that U only intersects Λ in the identity of G and define $\psi : \pi(U) \to \phi(U) : \pi(u) \mapsto \phi(u)$. Since G/Λ has a natural structure of complex analytic group (see *e.g.* [**48**, Corollary 2.6.2]) and $(\pi(U), \psi)$ obviously satisfies (i) and (ii) of Fact 2.7 (because (U, ϕ) satisfies both properties as it is a chart of G), by Fact 2.7 there exists $V \subset \pi(U)$ such that $\mathcal{A}_{(V,\psi)}$ is a $\mathbb{K}$-Nash atlas for G/Λ. When G/Λ is equipped with this locally $\mathbb{K}$-Nash structure, the projection π is a locally $\mathbb{K}$-Nash map. $\square$

5. Complex algebraic groups as locally $\mathbb{C}$-Nash groups

In this section, we check that there is a natural adaptation to our context of classical results concerning the relation between the algebraic and the analytic structure of complex algebraic groups.

Following the notation of [**38**], recall that an (abstract) algebraic group is a group $(G, \cdot)$ together with a finite covering of subsets $Y_1, \dots, Y_d$ of G with bijections f_i from Y_i onto an affine algebraic set X_i, for $i = 1, \dots, d$, such that

(1) For each pair (i, j), the set $X_{i,j} := f_i(Y_i \cap Y_j)$ is a Zariski open subset of X_i and $f_{i,j} := f_j \circ f_i^{-1}$ is locally rational. Recall that locally rational means that for each point of $X_{i,j}$ there exists a smaller Zariski open neighborhood of the point where the map is regular, that is, $f_{i,j}$ can be written as the quotient of two polynomials whose denominator does not vanish in that neighborhood.
(2) For each triple (i, j, k), both $\{(f_i(y_1), f_j(y_2)) \mid y_1y_2 \in Y_k\}$ and $\{f_i(y) \mid y^{-1} \in Y_j\}$ are respectively Zariski open subsets of $X_i \times X_j$ and X_i and the maps

$$\{(f_i(y_1), f_j(y_2)) \mid y_1y_2 \in Y_k\} \to X_k : (f_i(y_1), f_j(y_2)) \mapsto f_k(y_1y_2)$$

and

$$\{f_i(y) \mid y^{-1} \in Y_j\} \to X_j : f_i(y) \mapsto f_j(y^{-1})$$

are locally rational.

We call each pair (Y_i, f_i) a *Zariski chart* of G. A subset A of G is Zariski closed if $f_i(A \cap Y_i)$ is a Zariski closed subset of X_i for each $i = 1, ..., d$. We say that an (abstract) algebraic group is irreducible if its corresponding underlying variety is irreducible, that is, if it cannot be written as the union of two proper closed subsets.

We first show that a complex algebraic group has a natural structure of locally $\mathbb{C}$-Nash group. We recall that, given an affine algebraic set $X \subset \mathbb{C}^m$, the set of smooth points of X is

$$\mathrm{Smooth}(X) = \{a \in X \mid \dim(T_{X,a}) = \dim(X) := \min\{\dim(T_{X,b}) \mid b \in X\}\},$$

where $T_{X,a}$ denotes the Zariski tangent space. The set $\mathrm{Smooth}(X)$ is an open Zariski subset of X and it has a natural structure of complex analytic submanifold of $\mathbb{C}^m$. Specifically, if $X \subset \mathbb{C}^m$ is an n-dimensional affine algebraic set then for each $a \in \mathrm{Smooth}(X)$ there exist an open (in the strong topology of X coming from $\mathbb{C}^m$) subset $W_a \subset \mathrm{Smooth}(X)$ and a projection onto an open subset V_a on some coordinates $\pi_a : W_a \to V_a \subset \mathbb{C}^n$ such that π_a is a homeomorphism and the set of charts $\{(W_a, \pi_a)\}$ is an analytic atlas for $\mathrm{Smooth}(X)$. Moreover, the inverse $s_a := (\pi_a)^{-1} : V_a \to W_a \subset \mathbb{C}^m$ is analytic (see e.g. D. Mumford [**27**, Corollary 1.26] and the posterior paragraph).

In our case, given a complex algebraic group G, let $\mathrm{Smooth}(G)$ be the set of all preimages of the smooth points by the Zariski charts. Then, clearly $\mathrm{Smooth}(G) = G$. Indeed, given $g \in G$ and $h \in \mathrm{Smooth}(G)$, we consider the map $G \to G : x \mapsto gh^{-1}x$. Since this map is a rational isomorphism in the Zariski charts, we get that the image of h is also a smooth point, *i.e.*, that g is a smooth point. In particular, this shows that the images of all Zariski charts have the same dimension, say equal to n, and that G has a canonical n-dimensional complex manifold structure. *Henceforth, when we refer to open subsets of G we mean open with respect to this complex manifold structure.* Thus, we have the following corollary to Fact 2.7.

COROLLARY 2.15. *Let $(G, \cdot)$ be a n-dimensional complex algebraic group. Then, there exists a $\mathbb{C}$-Nash atlas $\mathcal{A}$ such that $(G, \cdot, \mathcal{A})$ is a locally $\mathbb{C}$-Nash group. Moreover, if $(Y, f = (f_1, \dots, f_m))$ is a Zariski chart of the identity $e \in G$, then there exist a projection $\pi : \mathbb{C}^m \to \mathbb{C}^n$ on some coordinates and an open subset $U \subset Y$ such that $\mathcal{A} = \mathcal{A}_{(U, \pi \circ f|_U)}$.*

PROOF. By the paragraph above, we already have an analytic structure on G, so it is enough to check properties (i) and (ii) of Fact 2.7. Let $f : Y \to X \subset \mathbb{C}^m$ be a Zariski chart of the identity $e \in Y \subset G$. Let $U \subset Y$ be an open neighborhood of e for which there exists a projection $\pi : \mathbb{C}^m \to \mathbb{C}^n$ on some coordinates such that $(U, \phi := \pi \circ f|_U)$ is a chart of the analytic structure of G. Note that we can assume that $f(U)$ is a semialgebraic set, so that $V := (\pi \circ f)(U)$ is also semialgebraic. In particular, the inverse $s : V \to f(U) \subset \mathbb{C}^m$ of the projection $\pi : f(U) \to V$ is semialgebraic and analytic, *i.e.*, a $\mathbb{C}$-Nash map.

Since G is an algebraic group, there exists an open neighborhood $U' \subset U$ of e such that

$$f \circ \cdot \circ (f^{-1}, f^{-1}) : f(U') \times f(U') \to f(U) \subset \mathbb{C}^m : (f(y_1), f(y_2)) \mapsto f(y_1 y_2)$$

satisfies that each coordinate function is the quotient of two polynomials whose denominator does not vanish. Without loss of generality, we can assume that $f(U')$ is semialgebraic, so the map above is a $\mathbb{C}$-Nash map. Finally, the map

$$\phi \circ \cdot \circ (\phi^{-1}, \phi^{-1})|_{\phi(U') \times \phi(U')} = \pi \circ (f \circ \cdot \circ (f^{-1}, f^{-1})) \circ (s, s)|_{\phi(U') \times \phi(U')}$$

is a composition of $\mathbb{C}$-Nash maps, so it is a $\mathbb{C}$-Nash map, as required.

Similarly, one shows that for each $g \in G$ there exists $U_g \subset U$ such that $\phi \circ -^g \circ \phi^{-1}|_{\phi(U_g)}$ is a $\mathbb{C}$-Nash map. To do this, one has to use that the conjugation $-^g$ is a rational map in the Zariski charts. □

We now analyze locally $\mathbb{C}$-Nash maps between algebraic groups (we use an idea which appears in [**20**]).

Corollary 2.16. *Let G_1 and G_2 be irreducible algebraic groups. If there exist an open semialgebraic subset W of the identity of G_1 and a locally $\mathbb{C}$-Nash map $g : W \to G_2$ that is a local isomorphism, then G_1 and G_2 are algebraically isogenous.*

Proof. We first note that both groups have the same dimension n as complex manifolds and, therefore, as algebraic groups. Let $f_i : Y_i \to X_i$ be a Zariski chart of the identity of G_i for $i = 1, 2$, where $Y_i \subset G_i$ and $X_i \subset \mathbb{C}^{m_i}$. We can assume that the image of the identity of G_i by f_i is 0. On the other hand, there exist open (euclidean) subsets V_i of X_i containing the identity for which there exist projections $\pi_i : V_i \to \mathbb{C}^n$ over some coordinates such that (V_i, π_i) is a chart of the locally $\mathbb{C}$-Nash structure of G_i. Without loss of generality, we can assume that the projections are over the first n coordinates.

Note that, shrinking W, we can assume that $g(W) \subset Y_2$. Moreover, we can assume that the open subset $U := (\pi_1 \circ f_1)(W)$ of $\mathbb{C}^n$ is contained in $\pi_1(V_1)$ and, therefore, $(\pi_2 \circ f_2) \circ g \circ (\pi_1 \circ f_1)^{-1}|_U = \pi_2 \circ (f_2 \circ g \circ f_1^{-1}) \circ \pi_1^{-1}|_U$ is a $\mathbb{C}$-Nash map.

In other words, if we denote $\widetilde{g} = f_2 \circ g \circ f_1^{-1}|_{\pi_1^{-1}(U)} = (\widetilde{g}_1, \ldots, \widetilde{g}_{m_2})$ then there exist polynomials $Q_j \in \mathbb{C}[\mathtt{z_1}, \ldots, \mathtt{z_n}, \mathtt{y}]$, $Q_j \neq 0$, such that

$$Q_j(z_1, \ldots, z_n, \widetilde{g}_j(\pi_1^{-1}(z_1, \ldots, z_n))) = 0$$

for $(z_1, \ldots, z_n) \in U$ and $j = 1, \ldots, n$. So, if we consider

$$Z_j = \{(z_1, \ldots, z_{m_1}, y_1, \ldots, y_{m_2}) \mid Q_j(z_1, \ldots, z_n, y_j) = 0\} \subset \mathbb{C}^{m_1} \times \mathbb{C}^{m_2}$$

and the Zariski closed subset $Z := (X_1 \times X_2) \cap \bigcap_{j=1}^{n} Z_j$ of $X_1 \times X_2$ then the graph of $\widetilde{g}$ is contained in Z. Furthermore, note that Z is n-dimensional near 0. Now, since the graph of $\widetilde{g}$ is an analytic manifold, we also deduce that Z must be irreducible. For, take two Zariski closed proper subsets Z'_1 and Z'_2 of Z such that $Z = Z'_1 \cup Z'_2$. Each $\mathrm{Graph}(\widetilde{g}) \cap Z'_i$ is an analytic subset of the connected analytic manifold $\mathrm{Graph}(\widetilde{g})$, so either they are equal to $\mathrm{Graph}(\widetilde{g})$ or they are thin in $\mathrm{Graph}(\widetilde{g})$. In the first case, we would get that $Z'_i = Z$ because Z is the Zariski closure of the graph, which contradicts the with properness. But the union of two thin sets is again thin, which is also a contradiction, as required.

All in all, we have shown that $A_W := \mathrm{Graph}(g|_W) \subset G_1 \times G_2$ is contained in its Zariski closure Z_W in $Y_1 \times Y_2$, which is a closed Zariski irreducible subset of $Y_1 \times Y_2 \subset G_1 \times G_2$ and n-dimensional near 0. Denote by $B_W := \overline{A_W}^{\mathrm{Zar}}$ the Zariski closure of A_W in $G_1 \times G_2$. Let us show that $B_W = \overline{Z_W}^{\mathrm{Zar}}$.

Indeed, as $A_W \subset Z_W$ we have $B_W \subset \overline{Z_W}^{\mathrm{Zar}}$. On the other hand, A_W is contained in the Zariski closed subset $B_W \cap (Y_1 \times Y_2)$ and, therefore, Z_W is also contained, so $\overline{Z_W}^{\mathrm{Zar}} \subset B_W$. Note that B_W is irreducible of dimension n near 0 because the closure of an irreducible set is irreducible and $B_W \cap (Y_1 \times Y_2) = \overline{Z_W}^{\mathrm{Zar}} \cap (Y_1 \times Y_2) = Z_W$.

Now, let $\mathcal{F}$ be the family of open (in the Euclidean topology) semialgebraic subsets W_1 of W such that $W_1 W_1 \subset W$ and $W_1 = W_1^{-1}$. For each $W_1 \in \mathcal{F}$, we consider the Zariski closure B_{W_1} of $A_{W_1} := \mathrm{Graph}(g|_{W_1}) \subset G_1 \times G_2$ in $G_1 \times G_2$, which is again irreducible and n-dimensional near 0. Let us denote $B = \bigcap_{W_1 \in \mathcal{F}} B_{W_1}$. This intersection is finite, so $B = B_{W_1}$ for some $W_1 \in \mathcal{F}$.

Let us see that B is a subgroup of $G_1 \times G_2$. Let $W_2 \in \mathcal{F}$, with $W_2 W_2 \subset W_1$, and note that $B_{W_2} = B_{W_1} = B$. Pick a point $a \in A_{W_2}$ and consider the Zariski closed subset

$$L_a = \{x \in G_1 \times G_2 \mid ax \in B\}.$$

Since $aA_{W_2} \subset A_{W_2} A_{W_2} \subset A_{W_1} \subset B$, we get $A_{W_2} \subset L_a$ and, therefore, $B \subset L_a$. On the other hand, consider the Zariski closed subset

$$R = \{a \in G_1 \times G_2 \mid aB \subset B\}.$$

We have showed that $A_{W_2} \subset R$, so that $B \subset R$. Finally, let us show that $B^{-1} = B$. Take the Zariski closed set $I = \{a \in G_1 \times G_2 \mid a^{-1} \in B\}$. Since $W_2^{-1} = W_2$, we have that $A_{W_2} \subset I$ and, therefore, $B \subset I$, as required.

We have proved that B is an irreducible algebraic subgroup of $G_1 \times G_2$ of dimension n near 0. As B is a group, it is actually connected and pure dimensional. Consider the projection $p_1 : B \to G_1$. Note that $p_1(B)$ is a n-dimensional subgroup of the irreducible G_1, so that p_1 is onto. We have that $\ker(p_1) = e \times F_2$, where F_2 is a subgroup of G_2. Moreover, since $\dim(B) = \dim(G_1) = n$, we deduce that F_2 is finite. Similarly, the projection $p_2 : B \to G_2$ is onto and $\ker(p_2) = F_1 \times e$, where F_1 is a finite subgroup of G_1. Therefore,

$$B/(F_1 \times F_2) = \big(B/(F_1 \times e)\big)/\big((F_1 \times F_2)/(F_1 \times e)\big) = G_2/F_2$$

and

$$B/(F_1 \times F_2) = \big(B/(e \times F_2)\big)/\big((F_1 \times F_2)/(e \times F_2)\big) = G_1/F_1,$$

where the above equalities occur in the algebraic category. Hence, we conclude that G_1 and G_2 are isogenous. □

Corollary 2.17. *Let G_1 and G_2 be irreducible algebraic groups. If G_1 and G_2 are locally $\mathbb{C}$-Nash isomorphic, then they are algebraically isomorphic.*

Proof. Let $g : G_1 \to G_2$ be a locally $\mathbb{C}$-Nash isomorphism. Let W be an small enough neighborhood of the identity of G_1 and consider the restriction $g|_W : W \to G_2$. As in the proof of Corollary 2.16, let B the irreducible subgroup of $G_1 \times G_2$ of dimension n that projects onto G_1 and G_2. Without loss of generality, we can assume that B is the Zariski closure of $A := \mathrm{Graph}(g|_W) \subset G_1 \times G_2$. Now, let H be the graph of g, which is an analytic connected subgroup of $G_1 \times G_2$ of dimension n. Since $H \cap B$ is an analytic subset of the analytic connected manifold H which contains an open set (because it contains A), it follows that $H \cap B = H$, *i.e.*, H is a subgroup of B. Since B is a connected group of dimension n and H is a n-dimensional subgroup, it follows that $B = H$. In particular, the isomorphism g is a morphism between algebraic varieties, as required. □

We finish this section pointing out that the category of locally Nash groups was first introduced as a natural setting where universal coverings of Nash groups and, in particular, of real algebraic groups could be defined. Later, E. Hrushovski and A. Pillay showed (see [**19, 20**]) that locally Nash groups fulfilled this role perfectly, since they are precisely quotients of universal coverings of real algebraic groups by discrete subgroups. Therefore, a classification of locally Nash groups will provide a first step towards a classification of Nash groups and of real algebraic groups.

The proof of Hrushovski and Pillay is based on Hrushovski's group configuration theorem [**18**], a model theoretic version of a classical result of A. Weil [**50**].

Fact 2.18 ([**19, 20**]). *Every simply connected n-dimensional abelian locally Nash group is the universal covering of the connected components of real algebraic groups of dimension n.*

In Chapter 3, we will prove the analogous statement for abelian locally $\mathbb{C}$-Nash groups, see Theorem 3.12.

Chapter 3

Abelian locally $\mathbb{K}$-Nash groups

In this chapter we show that abelian locally $\mathbb{K}$-Nash groups can be characterized via meromorphic maps admitting an algebraic addition theorem. We will also show in Theorem 3.12, using the results of Chapter 1 and a classical result of A. Weil [**50**], that simply connected n-dimensional abelian locally $\mathbb{C}$-Nash groups are exactly the universal coverings of the abelian complex algebraic groups.

The chapter is divided as follows. In Section 1, we recall some basic concepts of invariant meromorphic maps. In Section 2, we use the results of Chapters 1 and 2 to characterize abelian locally $\mathbb{K}$-Nash groups. Finally, in Section 3, we analyze meromorphic maps with discrete group of periods.

1. Invariant meromorphic maps

We will use the definitions and notations introduced in Chapter 1 (and specially at the beginning of its Section 1) and we recall that the elements of $\mathcal{M}_{\mathbb{C}^n}$ are the meromorphic functions.

We also recall that a meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^m$ is an *invariant meromorphic map* if $\overline{f(\overline{u})} = f(u)$ for each $u \in \mathbb{C}^n$ where f is defined and that a map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^m$ is $\mathbb{C}$-*meromorphic* if it is just meromorphic and $\mathbb{R}$-*meromorphic* if it is invariant meromorphic.

Let $U \subset \mathbb{K}^n$ be an open connected neighborhood of 0. Then $f_1, \dots, f_m \in \mathcal{M}_U$ are algebraically independent over $\mathbb{K}$ if and only if ${}^{\mathrm{t}}f_1, \dots, {}^{\mathrm{t}}f_m$ are algebraically independent over $\mathbb{K}$. Given $g = (g_1, \dots, g_m) \in \mathcal{M}_U^m$, we say that $f = (f_1, \dots, f_p) \in \mathcal{M}_U^p$ is *algebraic* over $\mathbb{K}(g) := \mathbb{K}(g_1, \dots, g_m)$ if each f_i is algebraic over $\mathbb{K}(g)$, equivalently ${}^{\mathrm{t}}f$ is algebraic over $\mathbb{K}({}^{\mathrm{t}}g)$.

We first recall some basic facts on invariant meromorphic functions (which can be generalized for invariant meromorphic maps in the obvious way).

Lemma 3.1. *Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be a meromorphic function. Then, there exists invariant meromorphic functions $Re(f), Im(f) : \mathbb{C}^n \dashrightarrow \mathbb{C}$ such that $f = Re(f) + iIm(f)$. If f is an analytic function then $Re(f)$ and $Im(f)$ are also analytic functions.*

Proof. Let

$$Re(f(u)) = \frac{f(u) + \overline{f(\overline{u})}}{2}, \quad Im(f(u)) = \frac{f(u) - \overline{f(\overline{u})}}{2i}.$$

By definition, $Re(f)$ and $Im(f)$ are invariant meromorphic functions such that $f = Re(f) + iIm(f)$. If f is an analytic function then, by definition, both $Re(f)$ and $Im(f)$ are analytic functions. $\square$

Lemma 3.2. *Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be a meromorphic function. Then, the following assertions are equivalent:*

(1) *f is an invariant meromorphic function.*
(2) *There exist invariant analytic functions $g, h : \mathbb{C}^n \to \mathbb{C}$, h not identically zero, such that $f = g/h$.*
(3) *${}^{\mathrm{t}}f \in \mathcal{M}_{\mathbb{R},n}$.*

Proof. (1) $\Longrightarrow$ (2) Since f is a meromorphic function, there exist analytic functions $G, H : \mathbb{C}^n \to \mathbb{C}$, $H \neq 0$, such that $f = \frac{G}{H}$. Since f is invariant,

$$f(u) = \frac{f(u) + \overline{f(\overline{u})}}{2} = \frac{1}{2}\left(\frac{G(u)}{H(u)} + \frac{\overline{G(\overline{u})}}{\overline{H(\overline{u})}}\right) = \frac{1}{2}\left(\frac{G(u)\overline{H(\overline{u})} + H(u)\overline{G(\overline{u})}}{H(u)\overline{H(\overline{u})}}\right)$$

So we take

$$g(u) := \frac{1}{2}(G(u)\overline{H(\overline{u})} + \overline{G(\overline{u})}H(u)), \quad h(u) := H(u)\overline{H(\overline{u})}.$$

It is easy to check that both g and h are invariant.

(2) $\Longrightarrow$ (3) Suppose first that f is an analytic function. Then, we may assume that f is invariant. We note that the coefficients of ${}^{\mathrm{t}}f$ can be computed using the different derivatives of f at 0. As $f(\mathbb{R}^n) \subset \mathbb{R}$, all derivatives of f at 0 are real numbers. Hence, ${}^{\mathrm{t}}f \in \mathcal{O}_{\mathbb{R},n}$, so we are done. Now, we do the general case. By hypothesis, there exist analytic functions $g, h : \mathbb{C}^n \to \mathbb{C}$, h not identically zero, such that $f = g/h$. We have already shown that ${}^{\mathrm{t}}g, {}^{\mathrm{t}}h \in \mathcal{O}_{\mathbb{R},n}$. Since ${}^{\mathrm{t}}h \neq 0$, we get that ${}^{\mathrm{t}}f \in \mathcal{M}_{\mathbb{R},n}$, so we are done.

(3) $\Longrightarrow$ (1) Suppose first that f is an analytic function. Let

$${}^{\mathrm{t}}f(z) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n},$$

be the Taylor power series expansion of $f(z_1, \dots, z_n)$ at 0. We note that

$${}^{\mathrm{t}}(\overline{f(\overline{z})}) = \sum_{\alpha \in \mathbb{N}^n} \overline{a_\alpha} z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

is the Taylor power series expansion of $\overline{f(\overline{z}_1, \dots, \overline{z}_n)}$ at 0. Since, by hypothesis, each $a_\alpha \in \mathbb{R}$, we get that $\overline{a_\alpha} = a_\alpha$ for all $\alpha \in \mathbb{N}^n$. Hence, ${}^{\mathrm{t}}f(z) = {}^{\mathrm{t}}(\overline{f(\overline{z})})$. Since both $f(z)$ and $\overline{f(\overline{z})}$ are analytic functions, this shows that $f(z) = \overline{f(\overline{z})}$, so we are done.

Suppose now that f is a meromorphic function with ${}^{\mathrm{t}}f \in \mathcal{M}_{\mathbb{R},n}$. Then, there exist analytic functions $g, h : \mathbb{C}^n \to \mathbb{C}$, h not identically zero, such that $f = g/h$ and ${}^{\mathrm{t}}g, {}^{\mathrm{t}}h \in \mathcal{O}_{\mathbb{R},n}$. Since g and h are analytic, we have already shown that $g(z) = \overline{g(\overline{z})}$ and $h(z) = \overline{h(\overline{z})}$, so

$$\overline{f(\overline{z})} = \frac{\overline{g(\overline{z})}}{\overline{h(\overline{z})}} = \frac{g(z)}{h(z)} = f(z),$$

as required. □

Invariant meromorphic maps will play a key role in the study of locally Nash groups. We will show that if a meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is algebraic over an invariant meromorphic map then its real and imaginary parts are algebraically related.

LEMMA 3.3. *Let $f_1, \dots, f_n : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be meromorphic functions. Let $g_1, \dots, g_n : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be invariant meromorphic functions which are algebraically independent over $\mathbb{C}$. If $g_1, \dots, g_n$ are algebraic over $\mathbb{C}(f_1, \dots, f_n)$, then:*

(1) *$\overline{f_1(\overline{u})}, \dots, \overline{f_n(\overline{u})}$ are algebraic over $\mathbb{C}(f_1(u), \dots, f_n(u))$.*
(2) *$\operatorname{tr.deg.}_{\mathbb{C}} \mathbb{C}(Re(f_1), \dots, Re(f_n), Im(f_1), \dots, Im(f_n)) = n$.*
(3) *$Re(f_i), Im(f_i)$ are algebraic over $\mathbb{C}(f_1, \dots, f_n)$, for $i = 1, \dots, n$.*

PROOF. We begin with (1). Taking into account transcendence degrees, it follows that $f_1, \dots, f_n$ are algebraic over $\mathbb{C}(g_1, \dots, g_n)$. Fix $k \in \{1, \dots, n\}$. Taking conjugation on the minimal polynomial of f_k over $\mathbb{C}(g_1, \dots, g_n)$ and substituting the variable u by $\bar{u}$, it is easy to see that $\overline{f_k(\overline{u})}$ is algebraic over $\mathbb{C}(\overline{g_1(\overline{u})}, \dots, \overline{g_1(\overline{u})})$. Since $g_1, \dots, g_n$ are invariant meromorphic functions, $\overline{f_k(\overline{u})}$ is algebraic over $\mathbb{C}(g_1, \dots, g_n)$ and, hence, over $\mathbb{C}(f_1, \dots, f_n)$.

We continue with (2). Fix $k \in \{1, \dots, n\}$. By the proof of Lemma 3.1, $2Re(f_k)(u) = f_k(u) + \overline{f_k(\overline{u})}$ and $2iIm(f_k)(u) = f_k(u) - \overline{f_k(\overline{u})}$, so $Re(f_k)$ and $Im(f_k)$ are algebraic over $\mathbb{C}(g_1, \dots, g_n)$. Hence, both $Re(f_1), \dots, Re(f_n)$ and $Im(f_1), \dots, Im(f_n)$ are algebraic over $\mathbb{C}(g_1, \dots, g_n)$. This shows that the transcendence degree of $\mathbb{C}(Re(f_1), \dots, Re(f_n), Im(f_1), \dots, Im(f_n)$ over $\mathbb{C}$ is at most n. In particular, its transcendence degree is exactly n because $f_1, \dots, f_n$ are algebraic over $\mathbb{C}(Re(f_1), \dots, Re(f_n), Im(f_1), \dots, Im(f_n))$.

For (3), note that $Re(f_1), \dots, Re(f_n), Im(f_1), \dots, Im(f_n)$ are algebraic over $\mathbb{C}(g_1, \dots, g_n)$ and $g_1, \dots, g_n$ are algebraic over $\mathbb{C}(f_1, \dots, f_n)$. □

The next property of the complex conjugation will be also useful.

LEMMA 3.4. *Let $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^n$ and $g : \mathbb{C}^n \dashrightarrow \mathbb{C}^p$ be meromorphic maps. Then, $\overline{g(f(\overline{u}))} = \overline{g(\overline{u})} \circ \overline{f(\overline{u})}$.*

PROOF. For each $i \in \mathbb{N}$, let $\sigma_i : \mathbb{C}^i \to \mathbb{C}^i : u \mapsto \overline{u}$ and note that $\sigma_i^2 = \mathrm{id}$. Then,

$$\overline{g(f(\overline{u}))} = \sigma_p(g(f(\sigma_m(u)))) = \sigma_p(g(\sigma_n(\sigma_n(f(\sigma_m(u)))))) = \overline{g(\overline{u})} \circ \overline{f(\overline{u})}.$$

□

2. Characterization of abelian locally $\mathbb{K}$-Nash groups

Let U be an open subset of $\mathbb{K}^n$. We will say $f \in \mathcal{M}_U^n$ admits an *algebraic addition theorem* if ${}^{\mathrm{t}}f \in \mathcal{M}_{\mathbb{K}^n,0}^n$ admits an AAT (see page 3). Not every element of $\mathscr{M}^n_{\mathbb{C},n}$ admitting an AAT comes from a meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an AAT. An example of this is the function $u \mapsto \sqrt{u+1}$ which, although it is not a meromorphic function, its Taylor power series expansion at 0 admits an AAT. We first rewrite Corollary 1.5 and Lemma 1.7 in terms of meromorphic functions.

Corollary 3.5. *Let $f, g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be $\mathbb{K}$-meromorphic maps such that f is algebraic over $\mathbb{K}(g)$. Then:*

(1) *If f admits an AAT then $f(u + a)$ is algebraic over $\mathbb{K}(f(u))$ for each $a \in \mathbb{K}^n$.*
(2) *If f admits an AAT then g admits an AAT. In particular, $f(u+a)$ admits an AAT, for each $a \in \mathbb{C}^n$.*

We recall that the only analytic structure on $(\mathbb{K}^n, +)$ is the standard one (the one given by the identity map, see e.g. [**48**, Corollary 2.13.3]) and that its compatible charts are exactly the analytic diffeomorphisms. In what follows, we will use these facts without further mention. Next, we relate AAT to properties of analytic groups, as mentioned before the proof of Fact 2.7.

Lemma 3.6. *Let (U, ϕ) be a chart of the identity of $(\mathbb{K}^n, +)$ compatible with its standard analytic structure. Then, the following are equivalent:*

(1) *There exists an open neighborhood of the identity $U' \subset U$ such that*
$$\phi \circ + \circ (\phi^{-1}, \phi^{-1}) : \phi(U') \times \phi(U') \to \phi(U) : (x, y) \mapsto \phi(\phi^{-1}(x) + \phi^{-1}(y))$$
is a $\mathbb{K}$-Nash map (and therefore by Fact 2.7, there exists an open neighborhood $V \subset U$ of 0 such that $(\mathbb{K}^n, +, \phi|_V)$ is a locally $\mathbb{K}$-Nash group).
(2) *$\phi \in \mathcal{O}_U$ admits an AAT.*

Proof. (1) implies (2): By hypothesis, $\phi(U')$ is semialgebraic, since it is the projection of the domain of a semialgebraic map. Fix $i \in \{1, \dots, n\}$. As we have mentioned in the definition of $\mathbb{K}$-Nash map, this hypothesis implies that there exists $P_i \in \mathbb{K}[X_1, \dots, X_{2n+1}]$, $P_i \neq 0$, such that
$$P_i(x_1, \dots, x_n, y_1, \dots, y_n, \phi_i(\phi^{-1}(x) + \phi^{-1}(y))) \equiv 0 \text{ on } \phi(U') \times \phi(U'),$$
where $x := (x_1, \dots, x_n)$ and $y := (y_1, \dots, y_n)$. Since ϕ is a diffeomorphism, letting $u := \phi^{-1}(x)$ and $v := \phi^{-1}(y)$, we deduce that
$$P_i(\phi_1(u), \dots, \phi_n(u), \phi_1(v), \dots, \phi_n(v), \phi_i(u + v)) \equiv 0 \text{ on } U' \times U'.$$
In addition, the coordinate functions $\phi_1, \dots, \phi_n$ are clearly algebraically independent. So ϕ admits an AAT.

(2) implies (1): Fix $i \in \{1, \dots, n\}$. If ϕ admits an AAT then there exists $P_i \in \mathbb{K}[X_1, \dots, X_{2n+1}]$, $P_i \neq 0$, such that
$$P_i(\phi_1(u), \dots, \phi_n(u), \phi_1(v), \dots, \phi_n(v), \phi_i(u + v)) \equiv 0 \text{ on } U' \times U'$$
for some open neighborhood of the identity $U' \subset U$. Since ϕ is a diffeomorphism, we can let $x := \phi(u)$ and $y := \phi(v)$ and argue as before once we shrink U to make it semialgebraic. □

We can now justify the notation of $(\mathbb{K}^n, +, f)$ given in the introduction for a locally $\mathbb{K}$-Nash group structure on $(\mathbb{K}^n, +)$. Indeed, if $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a $\mathbb{K}$-meromorphic map such that

(a) f admits an AAT, and
(b) there exist $a \in \mathbb{K}^n$ and an open neighborhood $U \subset \mathbb{K}^n$ of 0 such that
$$\varphi : U \to \mathbb{K}^n : u \mapsto \varphi(u) := f(u+a)$$
is an analytic diffeomorphism

then, by Lemma 3.6, there exists an open neighborhood $V \subset U$ of 0 such that $(\mathbb{K}^n, +, \varphi|_V)$ is a locally $\mathbb{K}$-Nash group.

It remains to check that the locally $\mathbb{K}$-Nash group structure is independent of a and the domains U and V, that is, we have to show that given a $\mathbb{K}$-meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an AAT and given $a_1, a_2 \in \mathbb{K}^n$ such that
$$\varphi_1 : U_1 \to \mathbb{K}^n : u \mapsto f(u+a_1), \quad \varphi_2 : U_2 \to \mathbb{K}^n : u \mapsto f(u+a_2),$$
satisfy condition (b) above, we have that $(\mathbb{K}^n, +, \varphi_1|_{V_1})$ and $(\mathbb{K}^n, +, \varphi_2|_{V_2})$ are isomorphic as locally $\mathbb{K}$-Nash groups (where $V_1 \subset U_1$ and $V_2 \subset U_2$ are given by Lemma 3.6). By Corollary 3.5, φ_1 is algebraic over $\mathbb{K}(\varphi_2)$. Hence, by Proposition 2.11 the identity map is a locally $\mathbb{K}$-Nash isomorphism between $(\mathbb{K}^n, +, \varphi_1|_{V_1})$ and $(\mathbb{K}^n, +, \varphi_2|_{V_2})$.

Remark 3.7. The precedent discussion and Corollary 3.5.(2) shows that if f is a $\mathbb{K}$-meromorphic map and f satisfies conditions (a) and (b) above, we can denote, without ambiguity, by $(\mathbb{K}^n, +, f)$ the locally $\mathbb{K}$-Nash group whose chart at 0 is given by a restriction of (a translate of) f.

In the rest of this memoir, when we write $(\mathbb{K}^n, +, f)$ is a locally $\mathbb{K}$-Nash group, we are also assuming that f satisfies properties (a) and (b) above.

This convention is useful, since now we can denote by $(\mathbb{K}, +, \wp_{\langle 1,i\rangle_{\mathbb{Z}}}(u))$ the locally $\mathbb{K}$-Nash group $(\mathbb{K}, +, \wp_{\langle 1,i\rangle_{\mathbb{Z}}}(u+a)|_U)$, where U is a sufficiently small neighborhood of the identity and $a \in \mathbb{K}$ is also sufficiently small. We note that without this convention the notation $(\mathbb{K}, +, \wp_{\langle 1,i\rangle_{\mathbb{Z}}}(u))$ would not make sense, since the map $\wp_{\langle 1,i\rangle_{\mathbb{Z}}}(u)$ is not even a local diffeomorphism at 0.

Now we are ready to prove the main results of this section.

Theorem 3.8. *Every simply connected n-dimensional abelian locally $\mathbb{K}$-Nash group is isomorphic to some $(\mathbb{K}^n, +, f)$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a $\mathbb{K}$-meromorphic map admitting an AAT.*

Proof. Let $(G, \cdot)$ be a simply connected n-dimensional abelian locally $\mathbb{K}$-Nash group equipped with a Nash atlas $\mathcal{B} := \{(W_i, \phi_i)\}_{i \in I}$. In particular, G is an analytic group with this atlas and, therefore, there exists an isomorphism of analytic groups
$$\alpha : (G, \cdot) \to (\mathbb{K}^n, +),$$
where $(\mathbb{K}^n, +)$ is equipped with its unique analytic group structure, the standard one (see e.g. [**48**, Corollary 2.13.3]). Since $\mathcal{B}$ is a $\mathbb{K}$-Nash atlas for $(G, \cdot)$, we have that
$$\mathcal{A} := \{\alpha(W_i), \phi_i \circ \alpha^{-1}\}_{i \in I}$$
is a $\mathbb{K}$-Nash atlas for $(\mathbb{K}^n, +)$ compatible with its standard analytic structure. Moreover, $(G, \cdot)$ equipped with $\mathcal{B}$ is clearly locally $\mathbb{K}$-Nash isomorphic to $(\mathbb{K}^n, +)$ equipped with $\mathcal{A}$ (see Proposition 2.11).

Now, consider a chart of the identity $(U,\phi) \in \mathcal{A}$. Firstly, note that, as analytic chart, (U,ϕ) must be compatible with the standard analytic structure of $(\mathbb{K}^n,+)$, so ϕ is an analytic diffeomorphism. Also, being a chart of a locally $\mathbb{K}$-Nash group structure, it satisfies condition (1) of Lemma 3.6, so ϕ admits an AAT. Now, we apply Theorem 1.11 to ${}^{\mathrm{t}}\phi$, the power series expansion of ϕ at 0, to obtain $\psi := (\psi_1,\ldots,\psi_n) \in \mathscr{M}_{\mathbb{K},n}^n$ convergent on $\mathbb{C}^n$ and admitting an AAT such that ${}^{\mathrm{t}}\phi$ is algebraic over $\mathbb{K}(\psi)$. Since ϕ is an analytic diffeomorphism, we deduce that

$$\phi_* : T_0U \to T_{\phi(0)}\phi(U)$$

is an isomorphism of vectorial spaces. Hence, $d({}^{\mathrm{t}}\phi_1),\ldots,d({}^{\mathrm{t}}\phi_n)$ are linearly independent over $\mathscr{M}_{\mathbb{K},n}$. In particular, by Lemma 1.13.(2), $d\psi_1,\ldots,d\psi_n$ are also linearly independent over $\mathscr{M}_{\mathbb{K},n}$.

Consider the $\mathbb{K}$-meromorphic map

$$f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n : u \mapsto f(u) := {}^{\mathrm{a}}\psi(u),$$

which admits an AAT by definition. We first show that there exists $a \in \mathbb{K}^n$ and an open neighborhood $U' \subset \mathbb{K}^n$ of 0 such that

$$\varphi : U' \to \mathbb{K}^n : u \mapsto \varphi(u) := f(u+a)$$

is an analytic diffeomorphism onto its image, so we will have a locally $\mathbb{K}$-Nash group structure $(\mathbb{K}^n,+,f)$. Shrinking U' if necessary, we may assume that (U',φ) is one of its charts. Indeed, by Lemma 1.3.(1), there exists an open dense subset $W \subset \mathbb{K}^n$ such that

$$f|_W : W \to \mathbb{K}^n$$

is analytic. Let Δ denote the determinant of the Jacobian of $f|_W$. Since $d\psi_1,\ldots,d\psi_n$ are linearly independent over $\mathscr{M}_{\mathbb{K},n}$, Δ is not identically zero on W, so there exists $a \in U \cap W$ such that $\Delta(a) \neq 0$. Thus, f is a local diffeomorphism at a, so there exists an open neighborhood $U' \subset \mathbb{K}^n$ of 0 such that $\varphi : U' \to \mathbb{K}^n : u \mapsto \varphi(u) := f(u+a)$ is a diffeomorphism onto its image, as required.

Finally, by Corollary 3.5, we have that f is algebraic over $\mathbb{K}(f(u+a))$ and, therefore, ϕ is algebraic over $\mathbb{K}(\varphi)$ on a sufficiently small open neighborhood of 0, so that, by Proposition 2.11, the identity map from $(\mathbb{K}^n,+,\phi|_U)$ to $(\mathbb{K}^n,+,\varphi|_{U'})$ is a locally $\mathbb{K}$-Nash isomorphism. □

Once Theorem 3.8 is proved, we may assume that every simply connected abelian locally $\mathbb{K}$-Nash group is of the form $(\mathbb{K}^n,+,f)$, for some $\mathbb{K}$-meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an AAT. The following result characterizes the isomorphisms between groups of this form.

Lemma 3.9. *Let $f,g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be $\mathbb{K}$-meromorphic maps admitting an AAT. Then, the locally $\mathbb{K}$-Nash groups $(\mathbb{K}^n,+,f)$ and $(\mathbb{K}^n,+,g)$ are isomorphic if and only if there exists $\alpha \in \mathrm{GL}_n(\mathbb{K})$ such that $g \circ \alpha$ is algebraic over $\mathbb{K}(f)$.*

Proof. By hypothesis, there exist $a_1 \in \mathbb{K}^n$ and an open neighborhood of the identity U of $\mathbb{K}^n$ such that $(\mathbb{K}^n,+,f)$ denotes $(\mathbb{K}^n,+,\phi|_U)$ where

$$\phi : U \to \mathbb{K}^n : u \mapsto f(u+a_1).$$

Similarly, there exist $a_2 \in \mathbb{K}^n$ and an open neighborhood of the identity V of $\mathbb{K}^n$ such that $(\mathbb{K}^n, +, g)$ denotes $(\mathbb{K}^n, +, \psi|_V)$ where

$$\psi : V \to \mathbb{K}^n : u \mapsto g(u + a_2).$$

By Corollary 3.5, $f_{u+a_1} := f(u + a_1)$ is algebraic over $\mathbb{K}(f(u))$ and vice versa. The same happens for $g_{u+a_2} := g(u + a_2)$ and $g(u)$. In particular, for any $\alpha \in \mathrm{GL}_n(\mathbb{K})$, the composition $g \circ \alpha$ is algebraic over $\mathbb{K}(f)$ if and only if $g_{u+a_2} \circ \alpha$ is algebraic over $\mathbb{K}(f_{u+a_1})$.

We suppose first that

$$\alpha : (\mathbb{K}^n, +, \phi|_U) \to (\mathbb{K}^n, +, \psi|_V)$$

is an isomorphism of locally $\mathbb{K}$-Nash groups. Note that $\alpha \in \mathrm{GL}_n(\mathbb{K})$. Applying Proposition 2.11, there exists $W \subset U \cap \alpha^{-1}(V)$ such that $\psi \circ \alpha$ is algebraic over $\mathbb{K}(\phi)$ on W. We deduce that $g_{u+a_2} \circ \alpha$ is algebraic over $\mathbb{K}(f_{u+a_1})$ and, therefore, $g \circ \alpha$ is algebraic over $\mathbb{K}(f)$.

We show the right to left implication. Since $g \circ \alpha$ is algebraic over $\mathbb{K}(f)$, it follows that $g_{u+a_2} \circ \alpha$ is algebraic over $\mathbb{K}(f_{u+a_1})$. Therefore, $\psi \circ \alpha$ is algebraic over $\mathbb{K}(\psi)$ on a sufficiently small neighborhood of 0. Finally, since α is a continuous isomorphism, we apply Proposition 2.11 to α to obtain that $(\mathbb{K}^n, +, \phi|_U)$ and $(\mathbb{K}^n, +, \psi|_V)$ are isomorphic. □

We now show that the classification of connected abelian locally $\mathbb{K}$-Nash groups reduces to the classification of quotients of $\mathbb{K}$-Nash structures over $(\mathbb{K}^n, +)$ by discrete subgroups.

Proposition 3.10. (I) *Every connected n-dimensional abelian locally $\mathbb{K}$-Nash group is isomorphic to some $(\mathbb{K}^n, +, f)/\Gamma$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a $\mathbb{K}$-meromorphic map admitting an AAT and Γ is a discrete subgroup of $(\mathbb{K}^n, +)$.*

(II) *Given discrete subgroups Γ_1 and Γ_2 of $(\mathbb{K}^n, +)$, two locally $\mathbb{K}$-Nash groups $(\mathbb{K}^n, +, \phi)/\Gamma_1$ and $(\mathbb{K}^n, +, \psi)/\Gamma_2$ are isomorphic if and only if there exists an isomorphism $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$ such that $\alpha(\Gamma_1) = \Gamma_2$.*

Proof. (I) Let G be a connected n-dimensional abelian locally $\mathbb{K}$-Nash group. By Proposition 2.13, $\widetilde{G}$ is a a simply connected n-dimensional abelian locally $\mathbb{K}$-Nash group, so we can apply Theorem 3.8.

(II) We begin with the left to right implication. Let π_1 and π_2 denote the projections of $(\mathbb{K}^n, +)$ onto $(\mathbb{K}^n, +)/\Gamma_1$ and $(\mathbb{K}^n, +)/\Gamma_2$, respectively. Let β be the locally $\mathbb{K}$-Nash isomorphism from $(\mathbb{K}^n, +, \phi)/\Gamma_1$ to $(\mathbb{K}^n, +, \psi)/\Gamma_2$. Take an open neighborhood of the identity $U \subset \mathbb{K}^n$ and an analytic section $s_2 : \beta(\pi_1(U)) \to \mathbb{K}^n$ such that $\pi_2 \circ s_2 = Id$ and $e \in s_2(\beta(\pi_1(U)))$. Then, the map $d_e(s_2 \circ \beta \circ \pi_1) : T_e(\mathbb{K}^n, +) \to T_e(\mathbb{K}^n, +)$ is a homomorphism of Lie algebras (since the Lie bracket is 0 because we are in the abelian case). By [**48**, Theorem 2.7.5], there exists a homomorphism of Lie groups, $\alpha : (\mathbb{K}^n, +) \to (\mathbb{K}^n, +)$, such that $d_e\alpha = d_e(s_2 \circ \beta \circ \pi_1)$. By symmetry, changing β by β^{-1}, we get that α is an isomorphism of Lie groups. So $\alpha \in Gl_n(\mathbb{K})$ and $\beta \circ \pi_1 = \pi_2 \circ \alpha$. Now, we note that both α and β are injective maps and, hence, $\ker(\beta \circ \pi_1) = \Gamma_1$, $\ker(\pi_2 \circ \alpha) = \alpha^{-1}(\Gamma_2)$ and $\alpha(\Gamma_1) = \Gamma_2$. It only remains to show that α is a locally $\mathbb{K}$-Nash map. By Proposition 2.13, the maps π_1 and s_2 are locally $\mathbb{K}$-Nash maps, so $\alpha|_U = s_2 \circ \beta \circ \pi_1|_U$ is a locally $\mathbb{K}$-Nash map. Thus, by Proposition 2.9, α is a locally $\mathbb{K}$-Nash homomorphism.

For the right to left implication, let

$$\beta : (\mathbb{K}^n, +)/\Gamma_1 \to (\mathbb{K}^n, +)/\Gamma_2 : u + \Gamma_1 \mapsto \alpha(u) + \Gamma_2.$$

By definition, β is an analytic isomorphism. It only remains to show that β is a locally $\mathbb{K}$-Nash map. Take a sufficiently small open neighborhood of the identity U of $(\mathbb{K}^n, +)/\Gamma_1$ and an analytic section $s_1 : U \to \mathbb{K}^n$ such that $\pi_1 \circ s_1 = Id$. We note that $\beta|_U = \pi_2 \circ \alpha \circ s_1|_U$. By Proposition 2.13, the maps s_1 and π_2 are locally $\mathbb{K}$-Nash maps, so $\beta|_U$ is a locally $\mathbb{K}$-Nash map. By Proposition 2.9, β is a locally $\mathbb{K}$-Nash map. □

Remark 3.11. Clause (II) in Proposition 3.10 is obviously equivalent to

(a) there exists an isomorphism $\alpha : (\mathbb{K}^n, +, \phi) \to (\mathbb{K}^n, +, \psi)$, and
(b) there exists an automorphism $\gamma : (\mathbb{K}^n, +, \psi) \to (\mathbb{K}^n, +, \psi)$ such that $\gamma(\Gamma_2) = \alpha(\Gamma_1)$.

Of course, the above rewriting of clause (II) is only useful if we have some information about the automorphism group of $(\mathbb{K}^n, +, \psi)$. In the two-dimensional complex case, we will compute in Proposition 4.31 all the relevant automorphism groups.

We now proof the corresponding complex case of that of Fact 2.18, in the abelian case.

Theorem 3.12. *Every simply connected n-dimensional abelian locally $\mathbb{C}$-Nash group is the universal covering of some (abstract) abelian complex algebraic group of dimension n.*

Proof. [*Initial preparation*] By Theorem 3.8, we may assume that the locally $\mathbb{C}$-Nash group is of the form $(\mathbb{C}^n, +, \phi)$, for some meromorphic map $\phi = (\phi_1, \ldots, \phi_n) : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an AAT. Moreover, by Theorem 1.11, we may assume that there exist a meromorphic function $\phi_0 : \mathbb{C}^n \dashrightarrow \mathbb{C}$ and $R \in \mathbb{C}[X_0, \ldots, X_n] \setminus \{0\}$, irreducible, such that $R(\phi_0, \ldots, \phi_n) = 0$ and, for each ϕ_i, $i \in \{0, \ldots, n\}$, there exist $R_i \in \mathbb{C}(X_0, \ldots, X_{2n+1})$ and $Q_i \in \mathbb{C}(X_0, \ldots, X_n)$ satisfying

$$\begin{aligned} \phi_i(u + v) &= R_i(\phi_0(u), \ldots, \phi_n(u), \phi_0(v), \ldots, \phi_n(v)) \quad \text{and} \\ \phi_i(-u) &= Q_i(\phi_0(u), \ldots, \phi_n(u)). \end{aligned}$$

Let κ be the field generated by the coefficients of $R, R_0, \ldots, R_n, Q_0, \ldots, Q_n$ and consider $V \subset \mathbb{C}^{n+1}$, the zero set of R. Our aim is to show that V is a pre-group in the sense of Weil [**50**], via the group $(\mathbb{C}^n, +, \phi)$. Let us denote by X the analytic subset of $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus X$ is the set of points where all $\phi_0, \ldots, \phi_n$ are defined and let us consider the natural map

$$\Phi : \mathbb{C}^n \setminus X \dashrightarrow V : u \mapsto (\phi_0(u), \ldots, \phi_n(u)).$$

We first note that, since $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{C}$, the affine irreducible algebraic subset V of $\mathbb{C}^{n+1}$ is n-dimensional.

[*Step 1*] Moreover, we can define the maps

$$\begin{aligned} f : V \times V &\dashrightarrow V \\ (x, y) &\mapsto (R_0(x, y), \ldots, R_n(x, y)) \end{aligned}$$

and

$$\begin{array}{rcl} g : V & \dashrightarrow & V \\ x & \mapsto & (Q_0(x), \ldots, Q_n(x)), \end{array}$$

which are rational maps over κ which satisfy the following properties:

(G1) If x, y are independent generic points of V over κ and $z = f(x, y)$ then

$$\kappa(x) \subset \kappa(z, y) \quad \text{and} \quad \kappa(y) \subset \kappa(x, z).$$

(G2) If x, y, t are independent generic points of V over κ then

$$f(f(x, y), t) = f(x, f(y, t)).$$

Indeed, take D the Zariski open subset of $V \times V$ of all points (x, y) where $f(f(x, y), g(x))$ is defined and consider the subset of D given by

$$\{(x, y) \in D \mid f(f(x, y), g(x)) = y\}.$$

The set is Zariski closed, so that it equals $D \cap Y$, where Y is a Zariski closed subset of $V \times V$. Moreover, since $D \cap Y$ contains $\Phi(\mathbb{C} \setminus X) \times \Phi(\mathbb{C} \setminus X)$, we deduce that Y has dimension $2n$ and, therefore, it equals the irreducible $V \times V$, so that $D \cap Y = D$. This shows that $\kappa(y) \subset \kappa(x, z)$ and, by symmetry, we get that $\kappa(x) \subset \kappa(z, y)$, as required. The proof of *G2* is similar. Hence, we have proved that V is a pre-group [**50**, §I.1].

[*Step 2*] By [**50**, Proposition 4], there exists a birational equivalence $\omega : V \dashrightarrow W$, where W is an affine variety group-chunk (see the definition just before [**50**, Proposition 3]). By [**50**, Theorem, p.375] and the beginning of the proof in [**50**, Section III.6], there exists an algebraic group G with a Zariski chart of the form $\rho : W_1 \to W$, such that:

(1) We have the following diagram:

$$\begin{tikzcd} \mathbb{C}^n \setminus X \arrow[d, "\Phi"] & W_1 \subset G \, . \arrow[d, shift left, "\rho"] \\ V \arrow[r, "\omega"] & W \arrow[u, dotted, bend left, "\rho^{-1}"] \end{tikzcd}$$

(2) The following equalities hold:

$$f(x, y) = (\rho^{-1} \circ \omega)^{-1}((\rho^{-1} \circ \omega)(x) \cdot (\rho^{-1} \circ \omega)(y))$$
$$g(x) = (\rho^{-1} \circ \omega)^{-1}([(\rho^{-1} \circ \omega)(x)]^{-1}),$$

for all points $x, y \in V$ on which the involved functions can be evaluated.

We also note that G is abelian because the algebraic subset of the irreducible $G \times G$ which consists in commuting elements does contain a subset of dimension $2n$.

[*Step 3*] Now, our purpose is to define a local isomorphism between $(\mathbb{C}^n, +)$ and G. Let Z be the algebraic subset of V for which the birational maps ω and g are well-defined maps in $V \setminus Z$. Consider also the analytic subset $\Phi^{-1}(Z)$ of $\mathbb{C}^n \setminus X$ and denote $U := (\mathbb{C}^n \setminus X) \setminus \Phi^{-1}(Z)$, which is an open dense subset of $\mathbb{C}^n \setminus X$. We also note that U is connected, since it is the complement in $\mathbb{C}^n$ of a (thin) analytic subset. Therefore, we can consider the following analytic map:

$$h := (\rho^{-1} \circ \omega) \circ \Phi : U \to W_1 \subset G.$$

[*Step 3.1*] We claim that the map h satisfies that $h(x+y) = h(x)h(y)$, for all $x, y \in U$. Indeed, the analytic subset $\{(x,y) \in U \times U : h(x+y) = h(x)h(y)\}$ has non-empty interior because of the above relation of f and the group operation of G. Since $U \times U$ is connected, they must coincide. Similarly, we have that $h(-x) = h(x)^{-1}$, for all $x \in U$.

[*Step 3.2*] As U is dense in $\mathbb{C}^n$ and the maps $\mathbb{C}^n \to \mathbb{C}^n : x \mapsto 2x$ and $\mathbb{C}^n \to \mathbb{C}^n : x \mapsto -x$ are homeomorphisms, the sets $\{x \in \mathbb{C}^n \mid 2x \in U\}$ and $\{x \in \mathbb{C}^n \mid -x \in U\}$ are dense subsets of $\mathbb{C}^n$. The subset of points of U where ϕ is a local analytic diffeomorphism is a proper analytic subset of U. Thus, there exists $a \in U$ such that $2a \in U$, $-a \in U$ and ϕ is a local diffeomorphism at a. In particular, there exists an open semialgebraic neighborhood U_1 of a for which we have that $U_1 + U_1 \subset U$ and such that $\phi|_{U_1}$ is an analytic diffeomorphism onto its image. Thus, consider the following analytic diffeomorphism onto its image,

$$\begin{array}{rcl} s : -a + U_1 & \to & G \\ -a + u & \mapsto & h(a)^{-1}h(u). \end{array}$$

[*Step 3.3*] We claim that s is a local isomorphism. Indeed, given points $u, v \in U_1$ for which $(-a+u)+(-a+v) \in -a+U_1$, we have that $-a+u+v \in U_1$ and, since $u+v \in U_1 + U_1 \subset U$, we deduce

$$\begin{aligned} s\big((-a+u)+(-a+v)\big) &= s\big(-a+(-a+u+v)\big) \\ &= h(a)^{-1}h(-a+u+v). \end{aligned}$$

As $-a \in U$ and $u+v \in U$, we deduce

$$\begin{aligned} s\big((-a+u)+(-a+v)\big) &= h(a)^{-1}\big(h(-a)h(u+v)\big) \\ &= h(a)^{-1}\big(h(-a)h(u)(v)\big) \\ &= h(a)^{-1}\big(h(a)^{-1}h(u)h(v)\big) \\ &= h(a)^{-1}h(u)h(a)^{-1}h(v) \\ &= s(-a+u)s(-a+v), \end{aligned}$$

as required.

[*Step 3.4*] By Corollary 2.15, we have that G has a locally $\mathbb{C}$-Nash structure and our aim now is to show that s is a locally $\mathbb{C}$-Nash map. We explicitly define charts of G and $(\mathbb{C}^n, +, \phi)$ to work with. Note that $h(a)^{-1}W_1 \to W : y \mapsto \rho(h(a)y)$ is also a Zariski chart. In fact, since $h(a) \in W_1$, it is a Zariski chart of the identity. By Corollary 2.15, there exist an open subset W_2 of W_1, with $h(a) \in W_2$, and a projection π such that

$$h(a)^{-1}W_2 \to \pi(\rho(W_2)) : y \mapsto \pi(\rho(h(a)y))$$

is a chart of the identity of the locally $\mathbb{C}$-Nash structure of G, as required. We also consider the following chart of 0 of $(\mathbb{C}^n, +, \phi)$,

$$-a + U_1 \to \phi(U_1) : y \mapsto \phi(a+y),$$

where we have shrunk U_1 so that $s(-a+U_1) \subset h(a)^{-1}W_2$.

To show that the map s is a locally $\mathbb{C}$-Nash map, using the above charts, it reduces to show that the map

$$\phi(U_1) \to \pi\big(\rho(W_2)\big) : z \mapsto \pi\big(\rho(h(\phi^{-1}(z)))\big)$$

is algebraic over $\mathbb{C}(\mathrm{id})$. By definition of the map h, we have that

$$\pi(\rho(h(\phi^{-1}(z)))) = \pi(\omega(\phi_0(\phi^{-1}(z)), z)).$$

On the other hand, ϕ_0 is algebraic over $\mathbb{C}(\phi)$ and, hence, $\phi_0 \circ \phi^{-1}$ is algebraic over $\mathbb{C}(\mathrm{id})$. Thus, since ω is a rational map and π is a projection, we deduce that s is a locally $\mathbb{C}$-Nash map, as required.

[*Step 3.5*] Finally, by [**48**, Theorem 2.8.2 and 2.7.5], the local isomorphism s can be lifted to an analytic global isomorphism $S : (\mathbb{C}^n, +, \phi) \to \widetilde{G}$. To show that S is a locally $\mathbb{C}$-Nash map it reduces to check that the map s is a locally $\mathbb{C}$-Nash (see Proposition 2.9), so we are done. □

3. Groups of periods of meromorphic maps

We introduce next some invariants that will allow us to describe $\mathbb{K}$-Nash atlas of $(\mathbb{K}^n, +)$ for the cases $n = 1$ and $n = 2$ in Chapters 4 and 5. Let Λ be a discrete subgroup of $(\mathbb{C}^n, +)$. Then, there exist $r \leq 2n$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$, linearly independent over $\mathbb{R}$, such that

$$\Lambda = \mathbb{Z}\lambda_1 \oplus \ldots \oplus \mathbb{Z}\lambda_r.$$

We call r the rank of Λ (the dimension of Λ as a free $\mathbb{Z}$-module), which is independent of the chosen basis, and denote it $\operatorname{rank} \Lambda$. A discrete subgroup Λ of $(\mathbb{C}^n, +)$ is a *lattice* if $\operatorname{rank} \Lambda = 2n$. We say that a subgroup $G < (\mathbb{C}^n, +)$ is an *invariant subgroup* if $G = \overline{G}$, *i.e.*, if $g \in G$ implies that $\overline{g} \in G$.

The previous concepts are related to meromorphic maps as follows. Given a meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^m$, we define the *group of periods of f* as

$$\Lambda_f := \{a \in \mathbb{C}^n \mid f(u) = f(u + a)\},$$

where $f(u) = f(u+a)$ means that if $f = g/h$ then $g(b)h(b+a) = h(b)g(b+a)$, for all $b \in \mathbb{C}^n$. Note that Λ_f is a subgroup of $(\mathbb{C}^n, +)$ that may not be discrete. However, we have the following:

Lemma 3.13. *Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be a meromorphic map. We have:*

1. *If f is a local diffeomorphism at 0, then Λ_f is a discrete subgroup of $(\mathbb{C}^n, +)$.*
2. *If f is a meromorphic map, then Λ_f is a subgroup of $(\mathbb{C}^n, +)$.*
3. *If f is an invariant meromorphic map, then Λ_f is an invariant subgroup of $(\mathbb{C}^n, +)$.*
4. *If f is a meromorphic map, $a \in \mathbb{C}^n$ and, for some open neighborhood of the identity $U \subset \mathbb{C}^n$, the restriction of $f(u + a)$ to U is an analytic diffeomorphism, then Λ_f is a discrete subgroup of $(\mathbb{C}^n, +)$.*
5. *If f is an invariant meromorphic map, $a \in \mathbb{R}^n$ and, for some open neighborhood of the identity $U \subset \mathbb{R}^n$, the restriction of $f(u + a)$ to U is an analytic diffeomorphism, then Λ_f is an invariant discrete subgroup of $(\mathbb{C}^n, +)$.*
6. *If Λ_f is a discrete subgroup of $(\mathbb{C}^n, +)$ and $\alpha \in \mathrm{GL}_n(\mathbb{C})$, then $\Lambda_{f \circ \alpha}$ is a discrete subgroup of $(\mathbb{C}^n, +)$ with $\operatorname{rank} \Lambda_{f\circ\alpha} = \operatorname{rank} \Lambda_f$.*

Proof. (1) Clearly Λ_f is a subgroup of $(\mathbb{C}^n, +)$. Suppose for a contradiction that Λ_f is not discrete. Then, there exists an infinite sequence

$\{a_k \mid k \in \mathbb{N}\}$ of points of Λ_f that converges to some $a \in \mathbb{C}^n$. Take $\epsilon > 0$ such that f is injective and analytic on an open ball of radius ϵ centered at 0, this can be done because f is a local diffeomorphism at 0. Take $N \in \mathbb{N}$ such that $\|a_k - a_N\| < \epsilon$ for all $k \geq N$. Since Λ_f is a subgroup of $(\mathbb{C}^n, +)$, $a_k - a_N \in \Lambda_f$ for all $k \in \mathbb{N}$. This implies that $f(a_k - a_N) = f(0)$ for all $k \in \mathbb{N}$, which contradicts that f is injective in the ball of radius ϵ centered at 0.

(2) By definition, Λ_f is a subgroup of $(\mathbb{C}^n, +)$.

(3) Fix $\lambda \in \Lambda_f$. By definition, $f(u) = f(u+\lambda)$. Hence, $f(\overline{u}) = f(\overline{u}+\overline{\lambda})$, because f is an invariant meromorphic function. Therefore, $\overline{\lambda} \in \Lambda_f$.

(4) We may assume that $a = 0$. By (1) and (2), Λ_f is a discrete subgroup of $(\mathbb{C}^n, +)$.

(5) We may assume that $a = 0$. Let J be the determinant of the Jacobian of $f|_U$ at 0. Since $f^{-1}|_{f(U)}$ exists, $J \neq 0$. Since the determinant of the Jacobian of f at 0 is also J, it is not 0. By the Inverse Mapping Theorem, f is a local diffeomorphism at 0. Hence, by (1) and (3), Λ_f is an invariant discrete subgroup of $(\mathbb{C}^n, +)$.

(6) Take $r \leq 2n$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ linearly independent over $\mathbb{R}$ such that $\Lambda_f = \mathbb{Z}\lambda_1 \oplus \ldots \oplus \mathbb{Z}\lambda_r$. Then, $\alpha^{-1}(\lambda_1), \ldots, \alpha^{-1}(\lambda_r) \in \Lambda$ are linearly independent over $\mathbb{R}$ and $\Lambda_{f\circ\alpha} = \mathbb{Z}\alpha^{-1}(\lambda_1) \oplus \ldots \oplus \mathbb{Z}\alpha^{-1}(\lambda_r)$. □

Remark 3.14. If $(\mathbb{R}^n, +, f)$ is a locally Nash group then $(\mathbb{C}^n, +, f)$ is a locally $\mathbb{C}$-Nash group. Indeed, assume that f satisfies conditions (a) and (b) of Remark 3.7 for $\mathbb{K} = \mathbb{R}$. Then, since the definition of AAT is independent of $\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$ (see page 3), f satisfies condition (a) for $\mathbb{K} = \mathbb{C}$. Also, by the proof of Lemma 3.13.(5), there exists a translate of f that is a local diffeomorphism at 0 and, hence, f satisfies (b) for $\mathbb{K} = \mathbb{C}$.

We proved in Theorem 3.8 that every locally $\mathbb{K}$-Nash group structure on $(\mathbb{K}^n, +)$ is of the form $(\mathbb{K}^n, +, f)$, for a certain $\mathbb{K}$-meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an AAT. In Proposition 3.17, we will show that $\operatorname{rank}\Lambda_f$ is invariant up to isomorphism. We first prove a technical lemma.

Lemma 3.15. *Let $f, g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be meromorphic maps such that Λ_f and Λ_g are discrete subgroups of $(\mathbb{C}^n, +)$. If g is algebraic over $\mathbb{C}(f)$ then there exists $N \in \mathbb{N} \setminus \{0\}$ such that $N\Lambda_f < \Lambda_g$ and, in particular,* $\operatorname{rank}\Lambda_f \leq \operatorname{rank}\Lambda_g$. *If, in addition, the coordinate functions of g are algebraically independent over $\mathbb{C}$ (as it happens when g admits an AAT) then* $\operatorname{rank}\Lambda_f = \operatorname{rank}\Lambda_g$.

Proof. We prove the first assertion in the statement. We may assume that $\Lambda_f \neq \{0\}$. Take $\lambda \in \Lambda_f \setminus \{0\}$ and fix $j \in \{1, \ldots, n\}$. Let $P_j(Z)$ be minimum polynomial of $g_j(u)$ over $\mathbb{C}(f(u))$. Since $\lambda \in \Lambda_f$, $P_j(g_j(u+k\lambda)) \equiv 0$ for each $k \in \mathbb{Z}$. Since $P_j(Z)$ has a finite number of roots, there exist $k_1, k_2 \in \mathbb{Z}$, $k_2 > k_1$, such that $g_j(u+k_1\lambda) = g_j(u+k_2\lambda)$. Let $N_j := k_2 - k_1 \in \mathbb{N}\setminus\{0\}$. Then, $g_j(u) = g_j(u+N_j\lambda)$, so $g_j(u) = g_j(u+kN_j\lambda)$, for each $k \in \mathbb{Z}$. Let N be the least common multiple of $N_1, \ldots, N_n$. Then, $g_j(u) = g_j(u+kN\lambda)$, for each $k \in \mathbb{Z}$ and each $j \in \{1, \ldots, n\}$, so $N\lambda \in \Lambda_g$. Let now $\{\lambda_1, \ldots, \lambda_m\}$ be a basis for Λ_f. Denote again by N the l.c.m. of the N's such that $N\lambda_i \in \Lambda_g$, for $i = 1, \ldots, m$. This N satisfies $N\lambda \in \Lambda_g$, for each $\lambda \in \Lambda_f$. This also shows that Λ_g contains at least $\operatorname{rank}\Lambda_f$ linearly independent vectors over

$\mathbb{R}$, so $\operatorname{rank}\Lambda_f \leq \operatorname{rank}\Lambda_g$. The other assertion follows by symmetry, since if $g_1,\dots,g_n$ are algebraically independent over $\mathbb{C}$ then f is algebraic over $\mathbb{C}(g)$. □

The next corollary of Lemma 3.15 will be useful to study Weierstrass $\wp$-functions in the context of the one-dimensional classification of locally $\mathbb{K}$-Nash groups.

COROLLARY 3.16. *Let $f,g:\mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be meromorphic maps such that both Λ_f and Λ_g are discrete subgroups of $\mathbb{C}^n$. If g is algebraic over $\mathbb{C}(f)$ then there exists a discrete subgroup Λ of $(\mathbb{C}^n,+)$ such that* $\operatorname{rank}\Lambda = \operatorname{rank}\Lambda_f$ *and both $\Lambda<\Lambda_f$ and $\Lambda<\Lambda_g$. Furthermore, if Λ_f is an invariant discrete subgroup then we can take Λ to be an invariant discrete subgroup.*

PROOF. By Lemma 3.15, there exists $N\in\mathbb{N}\setminus\{0\}$ such that $N\Lambda_f<\Lambda_g$. It suffices to take $\Lambda = N\Lambda_f$. □

PROPOSITION 3.17. *Let $(\mathbb{K}^n,+,f)$ and $(\mathbb{K}^n,+,g)$ be isomorphic locally $\mathbb{K}$-Nash groups, for some $\mathbb{K}$-meromorphic maps $f,g:\mathbb{C}^n\dashrightarrow\mathbb{C}^n$ that admit an AAT. If $(\mathbb{K}^n,+,f)$ and $(\mathbb{K}^n,+,g)$ are isomorphic then* $\operatorname{rank}\Lambda_f = \operatorname{rank}\Lambda_g$.

PROOF. By Lemma 3.9, there exists $\alpha\in \mathrm{GL}_n(\mathbb{K})$ such that $g\circ\alpha$ is algebraic over $\mathbb{K}(f)$. We note that, by Lemma 3.13.(4) and (5), both Λ_g and Λ_f are discrete subgroups of $(\mathbb{C}^n,+)$. By Lemma 3.13.(6), $\Lambda_{g\circ\alpha}$ is also a discrete subgroup of $(\mathbb{C}^n,+)$, with $\operatorname{rank}\Lambda_{g\circ\alpha}=\operatorname{rank}\Lambda_g$. Now, by Lemma 3.15, $\operatorname{rank}\Lambda_{g\circ\alpha}=\operatorname{rank}\Lambda_f$, so $\operatorname{rank}\Lambda_f=\operatorname{rank}\Lambda_g$. □

Lemma 3.15 leads us to the the following definition. Let $\mathbb{L}$ be a field of meromorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ of transcendence degree n over $\mathbb{C}$. Suppose that there exists $f:=(f_1,\dots,f_n):\mathbb{C}^n\dashrightarrow\mathbb{C}^n$ such that $\{f_1,\dots,f_n\}$ is a transcendence basis of $\mathbb{L}$ over $\mathbb{C}$ and Λ_f is a discrete subgroup of $(\mathbb{C}^n,+)$. Then, by Lemma 3.15, for all $g:=(g_1,\dots,g_n):\mathbb{C}^n\dashrightarrow\mathbb{C}^n$ such that $\{g_1,\dots,g_n\}$ is a transcendence basis of $\mathbb{L}$ over $\mathbb{C}$, we have that Λ_g is a discrete subgroup of $(\mathbb{C}^n,+)$ with $\operatorname{rank}\Lambda_g=\operatorname{rank}\Lambda_f$. Hence, we introduce the following notation, that will be useful in the proof of Theorem 4.28.

DEFINITION. Let $\mathbb{L}$ be a field of meromorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ of transcendence degree n over $\mathbb{C}$. Suppose that there exists $f:=(f_1,\dots,f_n):\mathbb{C}^n\dashrightarrow\mathbb{C}^n$ such that $\{f_1,\dots,f_n\}$ is a transcendence basis of $\mathbb{L}$ over $\mathbb{C}$ and Λ_f is a discrete subgroup of $(\mathbb{C}^n,+)$. Then, we say that $\mathbb{Z}$–$\operatorname{rank}\mathbb{L}=m$ if $\operatorname{rank}\Lambda_f=m$. Otherwise, we say that the $\mathbb{Z}$–rank of $\mathbb{L}$ is not defined. Let $\mathcal{P}=\{\mathbb{L}_\gamma \mid \gamma\in\Gamma\}$ be a family of fields $\mathbb{L}_\gamma$ of meromorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ of transcendence degree n over $\mathbb{C}$. We say that $\mathbb{Z}$–$\operatorname{rank}\mathcal{P}=m$ if $\mathbb{Z}$–$\operatorname{rank}\mathbb{L}_\gamma=m$ for every $\gamma\in\Gamma$.

Chapter 4

Abelian locally $\mathbb{C}$-Nash groups of dimensions 1 and 2

In this chapter we study abelian locally $\mathbb{C}$-Nash groups of dimensions one and two. We recall Weierstrass and Painlevé descriptions of one-variable and two-variable meromorphic maps admitting an AAT, respectively (Facts 4.2 and 4.10). By the results of Chapter 3, these meromorphic maps provide the charts of all the locally Nash structures on $(\mathbb{C},+)$ and $(\mathbb{C}^2,+)$. We study the period groups and the algebraicity relations of the mentioned charts to be able to distinguish isomorphism types of the corresponding groups. This will give us a classification of simply connected abelian locally Nash groups. Finally, we study their automorphism groups to get a description of abelian locally Nash groups, via Proposition 3.10 and Remark 3.11.

The chapter is divided as follows. In Section 1, we give a classification of one-dimensional simply connected locally $\mathbb{C}$-Nash groups (Theorem 4.7) and of their automorphism groups (Proposition 4.8), which leads to a classification of the general one-dimensional case (Theorem 4.9). In Section 2, we give a classification of two-dimensional simply connected abelian locally $\mathbb{C}$-Nash groups (Theorem 4.28) and of their automorphism groups (Proposition 4.31), which leads to a classification of the general abelian two-dimensional case (via Proposition 3.10 and Remark 3.11). We will also establish a bijection, via universal coverings, between two-dimensional abelian complex algebraic groups and two-dimensional simply connected abelian locally $\mathbb{C}$-Nash groups (Corollaries 4.29 and 4.30).

Since the charts of the locally $\mathbb{C}$-Nash groups which appear in the classifications are given in terms of Weierstrass elliptic functions, we will finish this introduction by recalling their definition and establishing some notations.

Notation 4.1. Given a lattice Ω of $(\mathbb{C},+)$, we will consider the Weierstrass functions σ_Ω, ζ_Ω and $\wp_\Omega$ (see e.g. K. Chandrasekharan [**11**, Ch.III and IV]). Recall that

$$\sigma_\Omega(u) = u \prod_{\omega\in\Omega\setminus\{0\}} \left(1-\frac{u}{\omega}\right)\exp\left(\frac{u}{\omega}+\frac{1}{2}\left(\frac{u}{\omega}\right)^2\right),$$

$$\zeta_\Omega(u) = \frac{\sigma'_\Omega(u)}{\sigma_\Omega(u)},$$
$$\wp_\Omega(u) = -\zeta'_\Omega(u)$$

and, given $\xi \in \mathbb{C}$, we denote

$$\widetilde{\sigma}_{\Omega,\xi}(u) := \frac{\sigma_\Omega(u-\xi)}{\sigma_\Omega(u)}.$$

Also, given $\omega \in \mathbb{C} \setminus \mathbb{R}$, we denote by $\wp_\omega$, ζ_ω, σ_ω, $\widetilde{\sigma}_{\omega,\xi}$ the functions, $\wp_{\langle 1,\omega\rangle_\mathbb{Z}}$, $\zeta_{\langle 1,\omega\rangle_\mathbb{Z}}$, $\sigma_{\langle 1,\omega\rangle_\mathbb{Z}}$ and $\widetilde{\sigma}_{\langle 1,\omega\rangle_\mathbb{Z},\xi}$ respectively.

1. One-dimensional locally $\mathbb{C}$-Nash groups

In this section, we give the classification of one-dimensional locally $\mathbb{C}$-Nash groups (Proposition 4.8, Theorem 4.7 and Theorem 4.9).

In Proposition 2.8, we have shown that any locally $\mathbb{C}$-Nash group is isomorphic to the same group structure but with its $\mathbb{C}$-Nash atlas constructed by translating one of its charts of the identity. Later, in Theorem 3.8, we have shown that any simply connected n-dimensional abelian locally $\mathbb{C}$-Nash group is isomorphic to $(\mathbb{C}^n, +, f)$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a meromorphic function admitting an AAT (see also Remark 3.7). Hence, the next step is to characterize the functions from $\mathbb{C}$ to $\mathbb{C}$ that admit an AAT. This was done by Weierstrass (see for example [**16**, Ch.VII]).

Fact 4.2 (Weierstrass). *If $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is a meromorphic function that admits an AAT, then there exists $\alpha \in \mathrm{GL}_1(\mathbb{C})$ such that f is algebraic over $\mathbb{C}(g \circ \alpha)$, where g is one of the following maps:*

(1) $g(u) = u$.
(2) $g(u) = e^u$.
(3) $g(u) = \wp_\Lambda(u)$, *for some lattice* $\Lambda < (\mathbb{C}, +)$,

where $\wp_\Lambda$ is the Weierstrass $\wp$-function associated to Λ.

Note that all the above functions admit an AAT (see e.g. [**11**, Ch.II] for basic properties of $\wp_\Lambda$).

By Theorem 3.8, Lemma 3.9 and Fact 4.2, any one-dimensional simply connected locally $\mathbb{C}$-Nash group is isomorphic either to $(\mathbb{C}, +, \mathrm{id})$ or $(\mathbb{C}, +, \exp)$ or $(\mathbb{C}, +, \wp_\Lambda)$, for some $\Lambda < (\mathbb{C}, +)$. It only remains to determine whether any two of the above locally $\mathbb{C}$-Nash structures are isomorphic. For this purpose, we prove some properties of the Weierstrass $\wp$-functions (and also of the Weierstrass ζ and σ functions, that will be used later in the two-dimensional case).

Lemma 4.3. *Let Ω_1 and Ω_2 be lattices of $(\mathbb{C}, +)$ such that $\Omega_1 < \Omega_2$ and $n \in \mathbb{N}$ be the index $[\Omega_2 : \Omega_1]$ of Ω_1 in Ω_2. Let $\xi \in \mathbb{C}$. Let $a_1, \ldots, a_n \in \Omega_2$ be such that $\Omega_2 = \bigcup_{i=1}^n (\Omega_1 + a_i)$. Then, there exist $\mathfrak{c}, C, C' \in \mathbb{C}$ such that:*

(1) $\wp'_{\Omega_2}(u) = \displaystyle\sum_{i=1}^n \wp'_{\Omega_1}(u + a_i)$.
(2) $\wp_{\Omega_2}(u) = \displaystyle\sum_{i=1}^n \wp_{\Omega_1}(u + a_i) - \mathfrak{c}$.

(3) $\zeta_{\Omega_2}(u) = \sum_{i=1}^{n} \zeta_{\Omega_1}(u + a_i) + \mathfrak{c}u + C.$

(4) $\sigma_{\Omega_2}(u) = e^{(\mathfrak{c}/2)u^2+Cu+C'} \prod_{i=1}^{n} \sigma_{\Omega_1}(u + a_i).$

(5) $\widetilde{\sigma}_{\Omega_2,\xi}(u) = e^{-\xi\mathfrak{c}u+(\mathfrak{c}/2)\xi^2-C\xi} \prod_{i=1}^{n} \widetilde{\sigma}_{\Omega_1,\xi}(u + a_i).$

Proof. We begin with the proof of (1). Recall that

$$\wp'_{\Omega_2}(u) = -2 \sum_{\omega\in\Omega_2} \frac{1}{(u-\omega)^3}.$$

So

$$\wp'_{\Omega_2}(u) = -2\sum_{i=1}^{n} \sum_{\omega\in\Omega_1} \frac{1}{(u-(\omega+a_i))^3}$$

and, hence, $\wp'_{\Omega_2}(u) = \sum_{i=1}^{n} \wp'_{\Omega_1}(u - a_i)$.

(2) is clear from (1). (3) We first recall that $\zeta'_\Omega = -\wp_\Omega$, for any lattice $\Omega < (\mathbb{C}, +)$. So there exists $C \in \mathbb{C}$ such that $\zeta_{\Omega_2}(u) = \sum_{i=1}^{n} \zeta_{\Omega_1}(u+a_i)+\mathfrak{c}u+C$. (4) We first recall that $(\ln(\sigma_\Omega(u)))' = \zeta_\Omega(u)$, for any lattice $\Omega < (\mathbb{C}, +)$. So there exists $C' \in \mathbb{C}$ such that $\sigma_{\Omega_2}(u) = e^{(\mathfrak{c}/2)u^2+Cu+C'} \prod_{i=1}^{n} \sigma_{\Omega_1}(u + a_i)$. (5) follows from (4). □

Remark 4.4. In the above lemma, the constant $\mathfrak{c}$ only depends on Ω_1 and Ω_2 and not on the $a_1, \ldots, a_n$ chosen. Hence, from now on, the constant $\mathfrak{c}$ will be denoted $\mathfrak{c}(\Omega_2, \Omega_1)$.

Lemma 4.5. *Let Λ_1 and Λ_2 be lattices of $(\mathbb{C}, +)$ such that $\Lambda_1 < \Lambda_2$. Then, $\wp_{\Lambda_1}$ and $\wp_{\Lambda_2}$ are algebraically dependent over $\mathbb{C}$.*

Proof. Since both Λ_1 and Λ_2 are lattices of $(\mathbb{C}, +)$, $\operatorname{rank}\Lambda_1 = \operatorname{rank}\Lambda_2$, and, hence, $[\Lambda_2 : \Lambda_1] < \infty$. Let $n = [\Lambda_2 : \Lambda_1]$. By Lemma 4.3, there exist $a_1, \ldots, a_n, C \in \mathbb{C}$ such that $\wp_{\Lambda_2}(u) = \sum_{i=1}^{n} \wp_{\Lambda_1}(u + a_i) + C$. By Corollary 3.5 and since $\wp_{\Lambda_1}$ admits an AAT, $\wp_{\Lambda_1}(u + a)$ is algebraic over $\mathbb{C}(\wp_{\Lambda_1}(u))$, for all $a \in \mathbb{C}$. So $\wp_{\Lambda_2}$ is algebraic over $\mathbb{C}(\wp_{\Lambda_1})$. □

Lemma 4.6. *Let Λ_1 and Λ_2 be lattices of $(\mathbb{C}, +)$ and let $\alpha \in \mathrm{GL}_1(\mathbb{C})$.*

(1) *$\wp_{\Lambda_1}$ and $\wp_{\Lambda_2}$ are algebraically dependent over $\mathbb{C}$ if and only if there exists a lattice $\Lambda < (\mathbb{C}, +)$ such that $\Lambda < \Lambda_1$ and $\Lambda < \Lambda_2$.*

(2) *$\wp_{\Lambda_1} \circ \alpha$ and $\wp_{\alpha^{-1}(\Lambda_1)}$ are algebraically dependent over $\mathbb{C}$.*

Proof. (1) The left to right implication is Corollary 3.15. The right to left implication is an application of Lemma 4.5.

(2) Since $\alpha \in \mathrm{GL}_1(\mathbb{C})$, there exists a unique $a \in \mathbb{C}^*$ such that $\alpha(u) = au$. The Weierstrass $\wp$-function verifies that $\wp_{\Lambda_1}(u) = b^2\wp_{b\Lambda_1}(bu)$, for all $b \in \mathbb{C}^*$ (see e.g. [**11**, Ch.III]). Hence, $(\wp_{\Lambda_1} \circ \alpha)(u) = \wp_{\Lambda_1}(au) = a^{-2}\wp_{a^{-1}\Lambda_1}(u)$. □

Theorem 4.7. (*I*) *Every simply connected one-dimensional locally $\mathbb{C}$-Nash group is isomorphic to a group of one and only one of the following types:*

(1) $(\mathbb{C}, +, \mathrm{id})$.

(2) $(\mathbb{C}, +, \exp)$.

(3) $(\mathbb{C},+,\wp_\omega)$, *for some* $\omega\in\mathbb{C}\setminus\mathbb{R}$.

(II) $(\mathbb{C},+,\wp_{\omega_1})$ *and* $(\mathbb{C},+,\wp_{\omega_2})$ *are isomorphic if and only* $\omega_2=\frac{a\omega_1+b}{c\omega_1+d}$ *for some* $a,b,c,d\in\mathbb{Z}$ *with* $ad-bc\neq 0$. *If such is the case, the isomorphisms are of the form* $\alpha(u)=\frac{nu}{c\omega_1+d}$, *for some* $n\in\mathbb{N}$.

Proof. *(I)* Firstly, we note that any connected one-dimensional analytic group (Lie group) is abelian (by [**48**, Corollary 2.13.3] and since one-dimensional complex Lie algebras are clearly abelian). Thus, by Theorem 3.8, any simply connected one-dimensional locally $\mathbb{C}$-Nash group is isomorphic to $(\mathbb{C},+,f)$, for some meromorphic function $f:\mathbb{C}\dashrightarrow\mathbb{C}$ admitting an AAT.

Moreover, f is algebraic over $\mathbb{C}(g\circ\alpha)$, for some $\alpha\in \mathrm{GL}_1(\mathbb{C})$, and g is either as in *(I)*, *(II)* or *(III)* of Fact 4.2. In addition, $\operatorname{rank}\Lambda_f=\operatorname{rank}\Lambda_g$ by Lemma 3.16. Clearly, if g is as in item (i) in Fact 4.2, then $\operatorname{rank}\Lambda_g=i-1$, so, by Corollary 3.9, none of the groups listed (1), (2) or (3) can be isomorphic. Furthermore:

Claim (1)*: If* $\operatorname{rank}\Lambda_f=0$, *then the identity map is an isomorphism from* $(\mathbb{C},+,f)$ *to* $(\mathbb{C},+,\mathrm{id})$. Indeed, in this case, $g=\mathrm{id}$. Take $a\in\mathbb{C}^*$ such that $\alpha(u)=au$. Since f is algebraic over $\mathbb{C}(\mathrm{id}\circ\alpha)$, f is algebraic over $\mathbb{C}(\mathrm{id})$, so, by Corollary 3.9, the identity map is an isomorphism from $(\mathbb{C},+,f)$ to $(\mathbb{C},+,\mathrm{id})$.

Claim (2)*: If* $\operatorname{rank}\Lambda_f=1$, *then the map* $\gamma(u)=2\pi i\omega_0^{-1}u$ *is an isomorphism from* $(\mathbb{C},+,f)$ *to* $(\mathbb{C},+,\exp)$, *where* $\omega_0\in\mathbb{C}^*$ *is such that* $\langle\omega_0\rangle_{\mathbb{Z}}=\Lambda_f$. In this case, $g=\exp$. Let $a\in\mathbb{C}^*$ be such that $\alpha(u)=au$. Then, $\Lambda_{\exp\circ\alpha}=\langle 2a^{-1}\pi i\rangle_{\mathbb{Z}}$. By Lemma 3.16, there exists $n\in\mathbb{N}^*$ such that $n\Lambda_{\exp\circ\alpha}<\Lambda_f$, so $a=\frac{2q\pi i}{\omega_0}$ for some $q\in\mathbb{Q}^*$. Since q is rational, for $\beta(u)=q^{-1}u$ we clearly have that $\exp\circ\beta$ is algebraic over $\mathbb{C}(\exp)$. Now, taking into account transcendence degrees and by the previous algebraic relations, $(\exp\circ\beta\circ\alpha)\circ\mathrm{id}$ is algebraic over $\mathbb{C}(f)$. Therefore, by Corollary 3.9, the identity map is an isomorphism from $(\mathbb{C},+,f)$ to $(\mathbb{C},+,\exp\circ\beta\circ\alpha)$. Finally, case (2) follows by noting that $\exp\circ\beta\circ\alpha=\exp(2\pi i\omega_0^{-1}u)=\exp\circ\gamma$.

Claim (3)*: If* $\operatorname{rank}\Lambda_f=2$, *then the identity map is an isomorphism from* $(\mathbb{C},+,f)$ *to* $(\mathbb{C},+,\wp_{\Lambda_f})$. In this case, $g=\wp_\Lambda$, for some lattice Λ of $(\mathbb{C},+)$. By Lemma 4.6.(2), f is algebraic over $\mathbb{C}(\wp_{\alpha^{-1}(\Lambda)})$. Since $\alpha^{-1}(\Lambda)$ is the group of periods of $\wp_{\alpha^{-1}(\Lambda)}$, by Lemma 3.16 there exists $n\in\mathbb{N}^*$ such that $n\alpha^{-1}(\Lambda)<\Lambda_f$. Denote $\Lambda'=n\alpha^{-1}(\Lambda)$. By Lemma 4.6.(1), $\wp_{\Lambda'}$ is algebraic over $\mathbb{C}(\wp_{\Lambda_f})$. On the other hand, since $\Lambda'<\alpha^{-1}(\Lambda)$, by Lemma 4.6.(1) we also get that $\wp_{\alpha^{-1}(\Lambda)}$ is algebraic over $\mathbb{C}(\wp_{\Lambda'})$. So f is algebraic over $\mathbb{C}(\wp_{\Lambda_f})$. This implies that $\wp_{\Lambda_f}\circ\mathrm{id}$ is algebraic over $\mathbb{C}(f)$ and, by Corollary 3.9, the identity map is an isomorphism from $(\mathbb{C},+,f)$ to $(\mathbb{C},+,\wp_{\Lambda_f})$.

Finally, case (3) follows taking into account that, for any pair $\omega_1,\omega_2\in\mathbb{C}^*$ that are linearly independent over $\mathbb{R}$, the map $\alpha(u)=\omega_1^{-1}u$ is an isomorphism from $(\mathbb{C},+,\wp_{\langle\omega_1,\omega_2\rangle_{\mathbb{Z}}})$ to $(\mathbb{C},+,\wp_{\omega_1^{-1}\omega_2})$. Indeed, let $\Lambda_1:=\langle\omega_1,\omega_2\rangle_{\mathbb{Z}}$ and $\Lambda_2:=\langle 1,\omega_1^{-1}\omega_2\rangle_{\mathbb{Z}}$. By Lemma 4.6.(2), $\wp_{\Lambda_2}\circ\alpha$ is algebraic over $\mathbb{C}(\wp_{\alpha^{-1}(\Lambda_2)})$. Since $\alpha^{-1}(\Lambda_2)=\Lambda_1$, α is the required isomorphism, by Corollary 3.9.

(II) For each $i \in \{1,2\}$, let $\Lambda_i = \langle 1, \omega_i\rangle_{\mathbb{Z}}$. We begin with the left to right implication. By Corollary 3.9, there exists $\alpha \in \mathrm{GL}_1(\mathbb{C})$ such that $\wp_{\Lambda_2} \circ \alpha$ is algebraic over $\mathbb{C}(\wp_{\Lambda_1})$. Let $\tau \in \mathbb{C}^*$ be such that $\alpha(u) = \tau u$. Let $\Lambda_2' := \alpha^{-1}(\Lambda_2) = \langle \tau^{-1}, \tau^{-1}\omega_2\rangle_{\mathbb{Z}}$, which is also the group of periods of $\wp_{\Lambda_2}\circ\alpha$. Since $\wp_{\Lambda_1}$ is algebraic over $\mathbb{C}(\wp_{\Lambda_2}\circ\alpha)$, by Lemma 3.16 there exists $n \in \mathbb{N}^*$ such that $n\Lambda_2' < \Lambda_1$. So $n\tau^{-1} \in \langle 1, \omega_1\rangle_{\mathbb{Z}}$ and, hence, $n\tau^{-1} = c\omega_1 + d$, for some $c, d \in \mathbb{Z}$. Also $n\tau^{-1}\omega_2 \in \langle 1,\omega_1\rangle_{\mathbb{Z}}$ and, hence, $\omega_2 = (a\omega_1+b)(c\omega_1+d)^{-1}$, for some $a, b, c, d \in \mathbb{Z}$. Furthermore, since $\omega_2 \notin \mathbb{R}$, we get that $a\omega_1 + b$ and $c\omega_1 + d$ are linearly independent over $\mathbb{R}$ and, hence, $ad - bc \neq 0$. Now, we prove the right to left implication. Given $n \in \mathbb{N}^*$ let $\alpha(u) = \frac{nu}{cw_1+d}$. Then, $n\alpha^{-1}(\Lambda_2) = \langle cw_1 + d, aw_1 + b\rangle_{\mathbb{Z}}$ is a sublattice of Λ_1. Hence, by Lemma 4.6.(1), $\wp_{\alpha^{-1}(\Lambda_2)}$ is algebraic over $\mathbb{C}(\wp_{\Lambda_1})$. Also, by Lemma 4.6.(2), we have that $\wp_{\Lambda_2}\circ\alpha$ is algebraic over $\mathbb{C}(\wp_{\alpha^{-1}(\Lambda_2)})$ and, hence, over $\mathbb{C}(\wp_{\Lambda_1})$. So, by Corollary 3.9, α is the required isomorphism. □

Finally, we state the classification of the general, not necessarily simply connected, case. Since there are not many cases to consider, we will give it explicitly via Remark 3.11. First, we must analyze the group of locally $\mathbb{C}$-Nash automorphisms of each one of the possible locally $\mathbb{C}$-Nash structures. Given a locally $\mathbb{C}$-Nash structure $(\mathbb{C}^n, +, \phi)$, we will denote by $\mathrm{Aut}(\mathbb{C}^n, +, \phi)$ the group of locally $\mathbb{C}$-Nash automorphisms of $(\mathbb{C}^n, +, \phi)$. Given $\omega \in \mathbb{C}\setminus\mathbb{R}$, we define $K_\omega := \mathbb{Q}(\omega)$ if ω is quadratic over $\mathbb{Q}$ and $K_\omega := \mathbb{Q}$ otherwise.

Proposition 4.8. *Let $\omega \in \mathbb{C}\setminus\mathbb{R}$. Then:*

(1) $\mathrm{Aut}(\mathbb{C}, +, \mathrm{id}) = \mathrm{GL}_1(\mathbb{C})$.
(2) $\mathrm{Aut}(\mathbb{C}, +, \exp) = \mathrm{GL}_1(\mathbb{Q})$.
(3) $\mathrm{Aut}(\mathbb{C}, +, \wp_\omega) = \mathrm{GL}_1(K_\omega)$.

Proof. In each case, $f = \mathrm{id}$, $f = \exp$ or $f = \wp_\Lambda$, by Corollary 3.9 we have that α is a locally $\mathbb{C}$-Nash automorphism if and only if $\alpha \in \mathrm{GL}_1(\mathbb{C})$ and $f\circ\alpha$ is algebraic over $\mathbb{C}(f)$. Hence, it is enough to check, in each case, for which $a \in \mathbb{C}^*$ the map $\alpha(u) = au$ has the property that $f\circ\alpha$ is algebraic over $\mathbb{C}(f)$. For (1), it obvious, and, for (2) and (3), we refer to W.D. Brownawell and K.K. Kubota [**9**, Corollary 5] and [**9**, Theorem 5], respectively (see also Fact 4.17). However, we point out that these are not difficult computations, using Theorem 4.7.(II) for (3). □

Theorem 4.9. (I) *Every connected one-dimensional locally $\mathbb{C}$-Nash group is isomorphic to one of the following:*

(1) $(\mathbb{C}, +, \mathrm{id})/\Gamma$, *where Γ is a discrete subgroup of $(\mathbb{C}, +)$.*
(2) $(\mathbb{C}, +, \exp)/\Gamma$, *where Γ is a discrete subgroup of $(\mathbb{C}, +)$.*
(3) $(\mathbb{C}, +, \wp_\omega)/\Gamma$, *where $\omega \in \mathbb{C}\setminus\mathbb{R}$ and Γ is a discrete subgroup of $(\mathbb{C}, +)$.*

(II) *None of these groups is isomorphic to one of a different type. Moreover:*

(i) *Two groups of the first type defined by the data Γ_1 and Γ_2, respectively, are isomorphic if and only if there exists $\alpha \in \mathrm{GL}_1(\mathbb{C})$ such that $\alpha(\Gamma_1) = \Gamma_2$.*

(*ii*) *Two groups of the second type defined by the data* Γ_1 *and* Γ_2*, respectively, are isomorphic if and only if there exists* $\alpha \in \mathrm{GL}_1(\mathbb{Q})$ *such that* $\alpha(\Gamma_1) = \Gamma_2$*.*

(*iii*) *Two groups of the third type defined respectively by the data* (ω_1, Γ_1) *and* (ω_2, Γ_2) *are isomorphic if and only if there exist* $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc \neq 0$ *such that* $\omega_2 = \frac{a\omega_1+b}{c\omega_1+d}$ *and there exists* $\gamma \in \mathrm{GL}_1(K_{\omega_2})$ *such that* $\gamma(\Gamma_2) = \alpha(\Gamma_1)$*, where*

$$\alpha : (\mathbb{C}, +, \wp_{\omega_1})/\Gamma_1 \to (\mathbb{C}, +, \wp_{\omega_2})/\Gamma_2 : u \mapsto \frac{u}{c\omega_1 + d}.$$

PROOF. It is enough to substitute (I) and (II) in Proposition 3.10 (and (a) and (b) of Remark 3.11) with the corresponding calculations of the one-dimensional case, namely Theorem 4.7 and Proposition 4.8. □

2. Two-dimensional abelian locally $\mathbb{C}$-Nash groups

We repeat the strategy followed in the one-dimensional case. The classification for simply connected locally $\mathbb{C}$-Nash groups will follow from a result of P. Painlevé, a two-dimensional analogue of Weierstrass result (Fact 4.2), and from a detailed study of the algebraicity of the Weierstrass functions. The abelian general, not necessarily simply connected, case will follow again from the computation of the automorphisms of the obtained simply connected groups.

We will divide this section in three parts: we will first recall Painlevé's description in Subsection 2.1, later, we will prove all the technical lemmas in Subsection 2.2 and, finally, we will give the proofs of the main results in Subsection 2.3.

2.1. Painlevé description of maps admitting an AAT. In this subsection we will recall a Theorem of Painlevé, published in [**31**]. Since Painlevé wrote [**31**] in 1902, some of its notation is outdated. We proceed to introduce and clarify its notation.

For Painlevé, a meromorphic map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admits an algebraic addition theorem if and only if the coordinate functions of f are functionally independent and f admits an AAT (in our sense). Any n functions are functionally independent if *ils ne sont liées par aucune relation identique*, see the footstep note of the first page of [**31**]. With this definition, Painlevé is referring to the classical functional independence, see for example W.F. Newns [**29**, Definition 3] for a detailed treatment. Another characterization of functional independence, which will be more convenient for our purposes, is the following (see [**29**, Proposition 1]). Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, we say that $f_1, \dots, f_n : \mathbb{K}^n \dashrightarrow \mathbb{K}$ are *functionally independent* if the image of $f := (f_1, \dots, f_n) : \mathbb{K}^n \dashrightarrow \mathbb{K}^n$ has an interior point in $\mathbb{K}^n$. We will apply Painlevé's results to meromorphic maps associated to translations of charts of the identity of locally $\mathbb{K}$-Nash groups, which are clearly functionally independent.

To state Painlevé's results, we introduce the following notation (see also [**45**, Ch. 5 §6]). Recall that a meromorphic function f is *degenerate* if its group of periods Λ_f is not a discrete subgroup of $\mathbb{C}^n$. Let Λ be a lattice of $(\mathbb{C}^n, +)$. We say a meromorphic function $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ is an *abelian function corresponding to* Λ if $\Lambda_f > \Lambda$. The abelian functions corresponding to Λ form a field that we denote $\mathbb{C}(\Lambda)$. Clearly, $\mathbb{C}(\Lambda)$ contains degenerate functions, for example all the constants. We say that $\mathbb{C}(\Lambda)$ is *non-degenerate* if it contains at least one function that is not degenerate. We note that, depending on Λ, the transcendence degree of $\mathbb{C}(\Lambda)$ can be from 0 to n. However, $\mathbb{C}(\Lambda)$ is non-degenerate if and only if its transcendence degree over $\mathbb{C}$ is n (see, [**45**, Ch. 5 §11 Theorems 5 and 6]).

Finally, we define the *families of the Painlevé's description* as follows:

$\mathcal{P}_1 := \{\, \mathbb{C}(g_1 \circ \alpha) \mid \alpha \in \mathrm{GL}_2(\mathbb{C}) \,\}$, where $g_1(u,v) := (u,v)$.

$\mathcal{P}_2 := \{\, \mathbb{C}(g_2 \circ \alpha) \mid \alpha \in \mathrm{GL}_2(\mathbb{C}) \,\}$, where $g_2(u,v) := (e^u, v)$.

$\mathcal{P}_3 := \{\, \mathbb{C}(g_3 \circ \alpha) \mid \alpha \in \mathrm{GL}_2(\mathbb{C}) \,\}$, where $g_3(u,v) := (e^u, e^v)$.

$\mathcal{P}_4 := \{\, \mathbb{C}(g_{4,\xi,\Omega} \circ \alpha) \mid \alpha \in \mathrm{GL}_2(\mathbb{C}),\ \xi \in \{0,1\},\ \Omega$ is a lattice of $(\mathbb{C},+) \,\}$,
where $g_{4,\xi,\Omega}(u,v) = (\wp_\Omega(u), v - \xi\zeta_\Omega(u))$.

$\mathcal{P}_5 := \{\, \mathbb{C}(g_{5,\xi,\Omega} \circ \alpha) \mid \alpha \in \mathrm{GL}_2(\mathbb{C}),\ \xi \in \mathbb{C},\ \Omega$ is a lattice of $(\mathbb{C},+) \,\}$,
where $g_{5,\xi,\Omega}(u,v) = \left(\wp_\Omega(u), \dfrac{\sigma_\Omega(u-\xi)}{\sigma_\Omega(u)} e^v\right)$.

$\mathcal{P}_6 := \{\, \mathbb{C}(\Lambda) \mid \Lambda$ is a lattice of $(\mathbb{C}^2,+)$, $\mathrm{tr.deg.}_{\mathbb{C}}\, \mathbb{C}(\Lambda) = 2\}$.

It can be checked that $g_{4,\xi,\Omega}$ is algebraic over $\mathbb{C}(g_{4,1,\Omega})$ for each $\xi \in \mathbb{C}^*$. This is the reason of why only $\xi \in \{0,1\}$ are considered in the family $\mathcal{P}_4$. Henceforth, we keep the notation g_1, g_2, g_3, $g_{4,\xi,\Omega}$ and $g_{5,\xi,\Omega}$ exclusively for these mentioned functions. We point out that g_1, g_2, g_3 clearly admit an AAT and $g_{4,\xi,\Omega}$ and $g_{5,\xi,\Omega}$ also admit an AAT by [**31**, Art. 16 and 19] and the fact that $g_{4,\xi,\Omega}$ is algebraic over $g_{4,1,\Omega}$ for all $\xi \in \mathbb{C}^*$. On the other hand, any transcendence basis of a field in $\mathcal{P}_6$ satisfies an AAT, by [**45**, Chap 5. §13]. Hence, all these maps induce locally $\mathbb{C}$-Nash group structures on $(\mathbb{C}^2, +)$.

Now, we can state the main result of [**31**].

Fact 4.10 (Painlevé, [**31**, Main Theorem]). *If $f_1, f_2 : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ are functionally independent meromorphic functions such that $f := (f_1, f_2)$ admits an AAT then there exist $i \in \{1, \ldots, 6\}$ such that $f_1(u,v)$ and $f_2(u,v)$ are algebraic over one of the fields of the family $\mathcal{P}_i$.*

In the next lemma, we analyze the group of periods of the families of Painlevé's theorem. Firstly, we will list the properties of the Weierstrass σ and ζ functions that will be needed.

Fact 4.11. ([**11**, Ch.IV]) *Let $\Omega := \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ be a lattice of $\mathbb{C}$. Then:*

(1) $\zeta_\Omega(z + \omega_i) = \zeta_\Omega(z) + 2\zeta_\Omega(\omega_i/2)$ *for each $i \in \{1,2\}$.*

(2) $\sigma_\Omega(z + \omega_i) = -\sigma_\Omega(z) e^{2\zeta_\Omega(\frac{\omega_i}{2})(z + \frac{\omega_i}{2})}$ *for each $i \in \{1,2\}$.*

Lemma 4.12. *Let $\xi \in \mathbb{C}$ and $\Omega := \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ be a lattice of $(\mathbb{C},+)$. Then:*

(1) $\Lambda_{g_1} = \{(0,0)\}$.

(2) $\Lambda_{g_2} = \langle (2\pi i, 0) \rangle_{\mathbb{Z}}$.

(3) $\Lambda_{g_3} = \langle (2\pi i, 0), (0, 2\pi i) \rangle_{\mathbb{Z}}$.

(4) $\Lambda_{g_{4,\xi,\Omega}} = \langle(\omega_1, 2\xi\zeta_\Omega(\omega_1/2)), (\omega_2, 2\xi\zeta_\Omega(\omega_2/2))\rangle_{\mathbb{Z}}$.
(5) $\Lambda_{g_{5,\xi,\Omega}} = \langle(\omega_1, 2\xi\zeta_\Omega(\omega_1/2)), (\omega_2, 2\xi\zeta_\Omega(\omega_2/2), (0, 2\pi i))\rangle_{\mathbb{Z}}$.

Proof. The only nontrivial cases are the last two ones, when $\xi \neq 0$. On the other hand, it is easy to check, using Fact 4.11, that the above tuples are periods of the corresponding map.

We begin with the case $g_{4,\xi,\Omega}$. Let g denote $g_{4,\xi,\Omega}$ and g_1 and g_2 denote the coordinate functions of g. Fix $\lambda := (\lambda_1, \lambda_2) \in \Lambda_g$. Clearly, $\Lambda_g \subset \Lambda_{g_1} \cap \Lambda_{g_2}$. Since $\lambda \in \Lambda_{g_1}$, we have $\lambda_1 \in \Omega$. Fix $m, n \in \mathbb{Z}$ such that $\lambda_1 = m\omega_1 + n\omega_2$. It follows from Fact 4.11.(1) that

$$\zeta_\Omega(u + m\omega_1 + n\omega_2) - \zeta_\Omega(u) = 2m\zeta_\Omega(\omega_1/2) + 2n\zeta_\Omega(\omega_2/2). \tag{4.1}$$

Since $\lambda \in \Lambda_{g_2}$, we deduce

$$v + \lambda_2 - \xi\zeta_\Omega(u + m\omega_1 + n\omega_2) = v - \xi\zeta_\Omega(u)$$

and, hence, by equation (4.1),

$$\lambda_2 = 2\xi m\zeta_\Omega(\omega_1/2) + 2\xi n\zeta_\Omega(\omega_2/2).$$

This means that the elements of Λ_g are of the form

$$(m\omega_1 + n\omega_2, 2\xi m\zeta_\Omega(\omega_1/2) + 2\xi n\zeta_\Omega(\omega_2/2)),$$

with $m, n \in \mathbb{Z}$, so we are done with this case.

Now we show the case $g_{5,\xi,\Omega}$. Let g denote $g_{5,\xi,\Omega}$ and g_1 and g_2 denote the coordinate functions of g. Fix $\lambda := (\lambda_1, \lambda_2) \in \Lambda_g$. Reasoning as before, there exists $m, n \in \mathbb{Z}$ such that $\lambda_1 = m\omega_1 + n\omega_2$. Moreover, again by Fact 4.11.(2) and from equation (4.1),

$$\frac{\sigma_\Omega(u + m\omega_1 + n\omega_2)}{\sigma_\Omega(u)} = Ce^{u(2m\zeta_\Omega(\omega_1/2) + 2n\zeta_\Omega(\omega_2/2))}, \tag{4.2}$$

for some constant $C \in \mathbb{C}$. Since $\lambda \in \Lambda_{g_2}$, we deduce

$$\frac{\sigma_\Omega(u + \lambda_1 - \xi)}{\sigma_\Omega(u + \lambda_1)} e^{v+\lambda_2} = \frac{\sigma_\Omega(u - \xi)}{\sigma_\Omega(u)} e^v.$$

Consequently,

$$e^{\lambda_2} = \frac{\sigma_\Omega(u - \xi)}{\sigma_\Omega(u)} \frac{\sigma_\Omega(u + \lambda_1)}{\sigma_\Omega(u + \lambda_1 - \xi)}.$$

So, by equation (4.2), we get

$$e^{\lambda_2} = e^{2\xi(m\zeta_\Omega(\omega_1/2) + n\zeta_\Omega(\omega_2/2))},$$

so

$$\lambda_2 = 2\xi m\zeta_\Omega(\omega_1/2) + 2\xi n\zeta_\Omega(\omega_2/2) + 2p\pi i,$$

for some $p \in \mathbb{Z}$. This means that the elements of Λ_g are of the form

$$(m\omega_1 + n\omega_2, 2\xi m\zeta_\Omega(\omega_1/2) + 2\xi n\zeta_\Omega(\omega_2/2) + 2p\pi i),$$

with $m, n, p \in \mathbb{Z}$, which concludes the proof. □

Now, we study the $\mathbb{Z}$–rank of the Painlevé's families of fields (the $\mathbb{Z}$–rank was defined in page 49).

Proposition 4.13. *The* $\mathbb{Z}$–rank*s of the Painlevé's families are the following:*

(1) $\mathbb{Z}\text{–rank}\,\mathcal{P}_1 = 0$.

(2) $\mathbb{Z}$–rank $\mathcal{P}_2 = 1$.
(3) $\mathbb{Z}$–rank $\mathcal{P}_3 = 2$.
(4) $\mathbb{Z}$–rank $\mathcal{P}_4 = 2$.
(5) $\mathbb{Z}$–rank $\mathcal{P}_5 = 3$.
(6) $\mathbb{Z}$–rank $\mathcal{P}_6 = 4$.

PROOF. Let $i \in \{1, 2, 3\}$. By Lemma 3.13.(6) and since $\alpha \in \mathrm{GL}_2(\mathbb{C})$, the fields belonging to the same family $\mathcal{P}_i$ have the same $\mathbb{Z}$–rank, which is $\operatorname{rank} \Lambda_{g_i}$. Then, we can apply Lemma 4.12 to deduce that $\operatorname{rank} \Lambda_{g_i} = i - 1$.

Let $i \in \{4, 5\}$. As above, it is enough to consider $\operatorname{rank} \Lambda_{g_{i,\xi,\Omega}}$. By Lemma 4.12, these ranks are independent of ξ and Ω, so $\operatorname{rank} \Lambda_{g_{i,\xi,\Omega}} = i - 2$.

Finally, we consider the case of the abelian functions. Let Λ be a lattice of $(\mathbb{C}^2, +)$ such that $\mathbb{C}(\Lambda)$ has transcendence degree 2 over $\mathbb{C}$. Fix a transcendence basis $\{f_1, f_2\}$ of $\mathbb{C}(\Lambda)$ and let us see that $\operatorname{rank} \Lambda_f = 4$. By definition, $\Lambda < \Lambda_f$ and, therefore, it is enough to check that Λ_f is discrete. Since $\operatorname{tr.deg.}_{\mathbb{C}} \mathbb{C}(\Lambda) = 2$, there exists a non-degenerate meromorphic function $g \in \mathbb{C}(\Lambda)$. In particular, g is algebraic over $\mathbb{C}(f)$. Arguing as in the proof of Lemma 3.15, if Λ_f is not discrete then Λ_g is not discrete, a contradiction. $\square$

2.2. Algebraicity of the charts with Weierstrass functions. We make here all the technical computations needed to clarify the possible isomorphisms in a fixed family of Theorem 4.28. The groups corresponding to the first until the sixth type are direct products of one-dimensional subgroups. Only the groups of the seventh and eighth type need a deeper study.

We give some identities that will be used extensively along this section (together with other basic properties of the Weierstrass functions that can be found in [**11**, Ch.III and IV]).

FACT 4.14. *Let Ω be a lattice of $(\mathbb{C}, +)$ and $\xi \in \mathbb{C}$. Then, for each $a \in \mathbb{C}^*$:*

(1) $\wp'_{a\Omega}(au) = a^{-3} \wp'_{\Omega}(u)$.
(2) $\wp_{a\Omega}(au) = a^{-2} \wp_{\Omega}(u)$.
(3) $\zeta_{a\Omega}(au) = a^{-1} \zeta_{\Omega}(u)$.
(4) $\sigma_{a\Omega}(au) = a\sigma_{\Omega}(u)$.
(5) $\widetilde{\sigma}_{a\Omega,a\xi}(au) = \widetilde{\sigma}_{\Omega,\xi}(u)$.

PROOF. (2) is [**11**, Ch.III, equation (2.2)]. (3) is [**11**, Ch.IV, equation (1.4)]. (4) is [**11**, Ch.IV, equation (2.6)]. (5) follows by definition of $\widetilde{\sigma}$ from (4). (1) follows taking the derivative in (2) with respect u. $\square$

Next, we begin the study of the groups of the seventh type of Theorem 4.28. Firstly, we show that the lattice that defines the chart can be taken of the required form. We will denote $\alpha(u, v) := (au + bv, cu + dv) \in \mathrm{GL}_2(\mathbb{C})$ if α is an analytic automorphism of $(\mathbb{C}^2, +)$, *i.e.*, if $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$.

LEMMA 4.15. *Let $\Omega := \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ be a lattice of $(\mathbb{C}, +)$ and $\xi \in \mathbb{C}^*$. Let $\tau := \omega_1^{-1}\omega_2$ and define $\alpha(u, v) := (\omega_1 u, \xi\omega_1^{-1} v) \in \mathrm{GL}_2(\mathbb{C})$. Then, $\big(\wp_{\Omega}(u), v - \xi\zeta_{\Omega}(u)\big) \circ \alpha$ is algebraic over $\mathbb{C}\big(\wp_{\tau}(u), v - \zeta_{\tau}(u)\big)$.*

PROOF. By Fact 4.14,

$$(\wp_\Omega(u), v - \xi\zeta_\Omega(u)) = (\omega_1^{-2}\wp_\tau(\omega_1^{-1}u), \omega_1^{-1}\xi(\omega_1\xi^{-1}v - \zeta_\tau(\omega_1^{-1}u))),$$

and we are done. □

Now, we analyze the isomorphism classes between the groups of the seventh type of Theorem 4.28 that satisfy a special relation. This is a preliminary step to prove the general case. We recall that, given lattices $\Omega_1 < \Omega_2$ of $(\mathbb{C}, +)$, the constant $\mathfrak{c}(\Omega_2, \Omega_1)$ was defined in Remark 4.4.

LEMMA 4.16. *Let Ω_1 and Ω_2 be lattices of $(\mathbb{C}, +)$ such that $\Omega_1 < \Omega_2$ and $[\Omega_2 : \Omega_1] = n$, for some $n \in \mathbb{N}^*$. Let $\mathfrak{c} := \mathfrak{c}(\Omega_2, \Omega_1)$. Let $\alpha(u, v) := (u, \mathfrak{c}u + nv)$. Then, $\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big) \circ \alpha$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)$.*

PROOF. Let $a_1, \dots, a_n \in \mathbb{C}$ such that $\Omega_2 = \bigcup_{i=1}^n (\Omega_1 + a_i)$. Let C be as in Lemma 4.3. Then, for $D := \sum_{i=1}^n n^{-1}\mathfrak{c}a_i - C$, we have that

$$v - \zeta_{\Omega_2}(u) = D + \sum_{i=1}^n \Big(n^{-1}v - n^{-1}\mathfrak{c}(u + a_i) - \zeta_{\Omega_1}(u + a_i)\Big).$$

Indeed, note that $D - \sum_{i=1}^n n^{-1}\mathfrak{c}(u + a_i) = -\mathfrak{c}u - C$ and, therefore, the equality follows from Lemma 4.3.(3).

Let $\beta(u, v) = (u, n^{-1}v - n^{-1}\mathfrak{c}u)$, so that $v - \zeta_{\Omega_2}(u) = D + \sum_{i=1}^n \big((v - \zeta_{\Omega_1}(u)) \circ \beta\big)(u + a_i, v)$. Since $\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)$ admits an AAT, clearly $\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big) \circ \beta$ also admits an AAT. By Corollary 3.5, we deduce that $\big((v - \zeta_{\Omega_1}(u)) \circ \beta\big)(u + a_i, v)$ is algebraic over $\big((\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)) \circ \beta\big)(u, v)$, for each $i \in \{1, \dots n\}$. So $v - \zeta_{\Omega_2}(u)$ is algebraic over $\mathbb{C}\big((\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)) \circ \beta\big)$. By Lemma 4.6.(1), $\wp_{\Omega_2}(u)$ is algebraic over $\wp_{\Omega_1}(u)$. As a function of two variables, $\wp_{\Omega_1}(u) = \big(\wp_{\Omega_1}(u)\big) \circ \beta$, so we also get that $\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big)$ is algebraic over $\mathbb{C}\big((\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)) \circ \beta\big)$. We are done because $\alpha = \beta^{-1}$. □

We have all the ingredients to study all the isomorphism classes of the groups of seventh type, which concludes the study of this case. To this aim, we need the following deep result concerning the algebraicity of Weierstrass functions:

FACT 4.17 ([**9**, Theorem 3]). *Fix $m \in \mathbb{N}$ and, for each $i \in \{1, \dots, m\}$, let $\Omega_i := \langle 1, \omega_i \rangle_{\mathbb{Z}}$, with $\omega_i \in \mathbb{C} \setminus \mathbb{R}$. Suppose that $\omega_j \neq (a + b\omega_i)(c + d\omega_i)^{-1}$ whenever $i \neq j$ and $a, b, c, d \in \mathbb{Z}$ with $ad - bc \neq 0$. Set $K_{m+1} := \mathbb{Q}$ and, for each $i \in \{1, \dots, m\}$, define $K_i := \mathbb{Q}(\omega_i)$ if ω_i is quadratic over $\mathbb{Q}$ and $K_i := \mathbb{Q}$ otherwise. For each $i \in \{1, \dots, m + 1\}$, let $a_{i,1}, \dots, a_{i,n_i} \in \mathbb{C}^*$. Then, the functions*

$$\{u, \wp_{\Omega_i}(a_{i,j}u), \zeta_{\Omega_i}(a_{i,j}u), \exp(a_{m+1,k}u) \mid i, j, k\},$$

where $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n_i\}$, $k \in \{1, \dots, n_{m+1}\}$, are algebraically independent over $\mathbb{C}$ if and only if the set $\{a_{\ell,1}, \dots, a_{\ell,n_\ell}\}$ is linearly independent over K_ℓ, for each $\ell \in \{1, \dots, m + 1\}$.

Even though it is not quite obvious, we will see in Proposition 4.26 and Proposition 4.27.(1) that Lemma 4.18 is actually a generalization of Lemma 4.16.

LEMMA 4.18. *Let $\Omega_1 := \langle 1, \omega_1\rangle_{\mathbb{Z}}$ and $\Omega_2\langle 1, \omega_2\rangle_{\mathbb{Z}}$, with $\omega_1, \omega_2 \in \mathbb{C}\setminus\mathbb{R}$. Then, there exist $a, b, c, d \in \mathbb{Z}$ such that $ad - bc \neq 0$ and $\omega_2 = \frac{a\omega_1+b}{c\omega_1+d}$ if and only if there exists $\alpha \in \mathrm{GL}_2(\mathbb{C})$ such that $\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big)\circ\alpha$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)$. Furthermore, if such is case, for such $n \in \mathbb{N}^*$, we may define*

$$\alpha(u,v) = \left(qu, q^{-1}\left(\mathfrak{c}(\Omega : n\Omega) - \frac{\mathfrak{c}(\Omega_1, n\Omega)[\Omega : n\Omega]}{[\Omega_1 : n\Omega]}\right)u + \frac{[\Omega : n\Omega]}{[\Omega_1 : n\Omega]}q^{-1}v\right),$$

where $q := \frac{n}{c\omega_1+d}$ and $\Omega = q^{-1}\Omega_2$.

PROOF. We begin with the left to right implication. Fix $n \in \mathbb{N}^*$ and let $\Omega := \langle \frac{c\omega_1+d}{n}, \frac{a\omega_1+b}{n}\rangle_{\mathbb{Z}}$. Let $\alpha_1(u,v) := (\frac{c\omega_1+d}{n}u, \frac{n}{c\omega_1+d}v)$. Then, by Lemma 4.15 (with $\xi = 1$), we get that $\big(\wp_\Omega(u), v - \zeta_\Omega(u)\big)\circ\alpha_1$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big)$. Clearly, $n\Omega < \Omega$, so that, by Lemma 4.16 for $\alpha_2(u,v) := \big(u, \mathfrak{c}(\Omega, n\Omega)u + [\Omega : n\Omega]v\big)$, we deduce $\big(\wp_\Omega(u), v - \zeta_\Omega(u)\big)\circ\alpha_2$ is algebraic over $\mathbb{C}\big(\wp_{n\Omega}(u), v - \zeta_{n\Omega}(u)\big)$.

We note also that $n\Omega < \Omega_1$, so let

$$\alpha_3(u,v) := \big(u, \mathfrak{c}(\Omega_1, n\Omega)u + [\Omega_1 : n\Omega]v\big).$$

By Lemma 4.16, it holds that $\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)\circ\alpha_3$ is algebraic over $\mathbb{C}\big(\wp_{n\Omega}(u), v - \zeta_{n\Omega}(u)\big)$. Consequently, $\big(\wp_\Omega(u), v - \zeta_\Omega(u)\big)$ is algebraic over $\mathbb{C}\big(\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)\circ\alpha_3\circ\alpha_2^{-1}\big)$. In particular, $\mathbb{C}\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big)$ is algebraic over $\mathbb{C}\big(\big(\wp_{\Omega_1}(u), v - \zeta_{\Omega_1}(u)\big)\circ\alpha_3\circ\alpha_2^{-1}\circ\alpha_1\big)$. Finally, note that $\alpha = (\alpha_3\circ\alpha_2^{-1}\circ\alpha_1)^{-1}$.

For the converse, let $\alpha(u,v) = (a'u+b'v, c'u+d'v) \in \mathrm{GL}_2(\mathbb{C})$. If we show that $a' \neq 0$ and $\wp_{\Omega_2}(a'u)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u))$ then, by Lemma 3.9, the function $u \mapsto a'u$ is an isomorphism between $(\mathbb{C}, +, \wp_{\Omega_1})$ and $(\mathbb{C}, +, \wp_{\Omega_2})$ and, hence, the lemma follows from Theorem 4.7.(II). So it is enough to prove the following more general result (that will be also used in the proof of Theorem 4.28):

Claim. Let $\alpha(u,v) = (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$. Fix $\xi \in \{0,1\}$. If $\big(\wp_{\Omega_2}(u), v - \zeta_{\Omega_2}(u)\big)\circ\alpha$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_1}(u), v - \xi\zeta_{\Omega_1}(u)\big)$ then $a \neq 0$, $b = 0$ and $\wp_{\Omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u))$.

Proof of the claim. Suppose that $b \neq 0$. Assume first, for a contradiction, that also $a \neq 0$. We recall that $\wp_{\Omega_2}$ admits an AAT, so $\wp_{\Omega_2}(au+bv)$ is algebraic over $\mathbb{C}(\wp_{\Omega_2}(au), \wp_{\Omega_2}(bv))$. Hence, $\wp_{\Omega_2}(bv)$ is algebraic over $\mathbb{C}(\wp_{\Omega_2}(au+bv), \wp_{\Omega_2}(au))$. Thus, $\wp_{\Omega_2}(bv)$ is also algebraic over

$$\mathbb{C}\left(\wp_{\Omega_2}(au+bv), cu+dv - \zeta_{\Omega_2}(au+bv), \wp_{\Omega_2}(au)\right).$$

By hypothesis, we also have that

$$(\wp_{\Omega_2}(au+bv), cu+dv - \zeta_{\Omega_2}(au+bv))$$

is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), v - \xi\zeta_{\Omega_1}(u))$, so $\wp_{\Omega_2}(bv)$ is algebraic over

$$\mathbb{C}\left(\wp_{\Omega_1}(u), v - \xi\zeta_{\Omega_1}(u), \wp_{\Omega_2}(au)\right).$$

Since $\wp_{\Omega_2}(bv)$ only depends on v, we get that $\wp_{\Omega_2}(bv)$ is algebraic over $\mathbb{C}(v)$. This contradicts Fact 4.17. If $a = 0$ then we obtain a similar contradiction, because $\wp_{\Omega_2}(bv)$ would be algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), v - \xi\zeta_{\Omega_1}(u))$. Both contradictions imply that $b = 0$ and, since $\alpha \in GL_2(\mathbb{C})$, that $a \neq 0$.

As $b=0$, we conclude that $\wp_{\Omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), v-\xi\zeta_{\Omega_1}(u))$. If $\wp_{\Omega_2}(au)$ is not algebraic over $\mathbb{C}(\wp_{\Omega_1}(u))$, then $v-\xi\zeta_{\Omega_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), \wp_{\Omega_2}(au))$, which is impossible, because this field does not depend on the variable v. □

The study of the groups of the eighth type in Theorem 4.28 is similar to the one of the groups of the seventh type, except that we cannot normalize simultaneously both parameters Ω and ξ (as we have done in Lemma 4.15 for the seventh type). We recall that $\widetilde{\sigma}_{\Omega,\xi}(u)=\frac{\sigma_\Omega(u-\xi)}{\sigma_\Omega(u)}$.

Lemma 4.19. *Let $\Omega:=\langle\omega_1,\omega_2\rangle_{\mathbb{Z}}$ be a lattice of $(\mathbb{C},+)$ and $\xi\in\mathbb{C}$. Let $\tau:=\omega_1^{-1}\omega_2$. Let $\alpha(u,v):=(\omega_1 u, v)$. Then, $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))\circ\alpha$ is algebraic over $\mathbb{C}(\wp_\tau(u), e^v\widetilde{\sigma}_{\tau,\omega_1^{-1}\xi}(u))$.*

Proof. By Fact 4.14,

$$(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)) = \left(\omega_1^{-2}\wp_\tau(\omega_1^{-1}u), e^v\widetilde{\sigma}_{\tau,\omega_1^{-1}\xi}(\omega_1^{-1}u)\right)$$

and we are done □

Now, we analyze the isomorphism classes between groups of the eighth type in Theorem 4.28 that satisfy a special relation. This is a preliminary step to prove the general case.

Lemma 4.20. *Let Ω_1 and Ω_2 be lattices of $(\mathbb{C},+)$ such that $\Omega_1<\Omega_2$ and $[\Omega_2:\Omega_1]=n$, for some $n\in\mathbb{N}^*$. Let $\mathfrak{c}:=\mathfrak{c}(\Omega_2,\Omega_1)$, $\xi\in\mathbb{C}$ and $\alpha(u,v):=(u,\xi\mathfrak{c}u+nv)$. Then, $(\wp_{\Omega_2}(u), e^v\widetilde{\sigma}_{\Omega_2,\xi}(u))\circ\alpha$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))$.*

Proof. Let $a_1,\dots,a_n\in\mathbb{C}$ be such that $\Omega_2=\bigcup_{i=1}^n(\Omega_1+a_i)$. Let C be as in Lemma 4.3 and let $D:=e^{(\mathfrak{c}/2)\xi^2-C\xi}\prod_{i=1}^n e^{n^{-1}\xi\mathfrak{c}a_i}$. Note that,

$$e^{-\xi\mathfrak{c}u+(\mathfrak{c}/2)\xi^2-C\xi} = D\prod_{i=1}^n(e^{-n^{-1}\xi\mathfrak{c}(u+a_i)}),$$

so, by Lemma 4.3.(5),

$$e^v\widetilde{\sigma}_{\Omega_2,\xi}(u) = D\prod_{i=1}^n\left(e^{n^{-1}v-n^{-1}\xi\mathfrak{c}(u+a_i))}\widetilde{\sigma}_{\Omega_1,\xi}(u+a_i)\right).$$

Take $\beta(u,v):=(u,n^{-1}v-n^{-1}\xi\mathfrak{c}u)$, so that

$$e^v\widetilde{\sigma}_{\Omega_2,\xi}(u) = D\prod_{i=1}^n((e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta)(u+a_i,v).$$

Since $(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))$ admits an AAT, $(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta$ also admits an AAT. By Corollary 3.5, we get that $((e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta)(u+a_i,v)$ is algebraic over $((\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta)(u,v)$, for each $i\in\{1,\dots n\}$. So $e^v\widetilde{\sigma}_{\Omega_2,\xi}(u)$ is algebraic over $\mathbb{C}((\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta)$. By Lemma 4.6.(1), $\wp_{\Omega_2}(u)$ is algebraic over $\wp_{\Omega_1}(u)$. We note that, as a function of two variables, $\wp_{\Omega_1}(u)=(\wp_{\Omega_1}(u))\circ\beta$. So we also have that $(\wp_{\Omega_2}(u), e^v\widetilde{\sigma}_{\Omega_2,\xi}(u))$ is algebraic over $\mathbb{C}((\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi}(u))\circ\beta)$. Now, the Lemma follows noting that $\alpha=\beta^{-1}$. □

Since parameters Ω and ξ cannot be simultaneously normalized, we choose to normalize Ω, so we have to analyze whether or not there exists an isomorphism between two groups with the same lattice Ω but different parameters ξ_1 and ξ_2. To this end, for each lattice Ω of $(\mathbb{C},+)$ and each $\xi \in \mathbb{C}$, we define

$$\begin{aligned}\Xi(\Omega,\xi) := \{z \in \mathbb{C} : & \text{ there exists } \alpha \in \mathrm{GL}_2(\mathbb{C}) \text{ such that} \\ & (\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,z}(u)) \circ \alpha \text{ is algebraic over } \mathbb{C}\left(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)\right)\}.\end{aligned}$$

As a last step before proving the general case, we have to study the properties of the set $\Xi(\Omega,\xi)$. We recall that given $\omega \in \mathbb{C}\setminus\mathbb{R}$, we have defined $K_\omega = \mathbb{Q}(\omega)$ if ω is quadratic over $\mathbb{Q}$ and $K_\omega = \mathbb{Q}$ otherwise. Also, given $\omega \in \mathbb{C}\setminus\mathbb{R}$, we denote by $\Xi(\omega,\xi)$ the set $\Xi(\langle 1,\omega\rangle_{\mathbb{Z}},\xi)$.

Lemma 4.21. *Let Ω be a lattice of $(\mathbb{C},+)$ and $\xi,\xi' \in \mathbb{C}$. We have:*

(1) *"$\xi \sim_\Omega \xi'$ if and only if $\xi' \in \Xi(\Omega,\xi)$" is an equivalence relation.*
(2) *If Ω' is a sublattice of Ω, then $\Xi(\Omega',\xi) = \Xi(\Omega,\xi)$.*
(3) *If $\wp_\Omega(\xi'\xi^{-1}u)$ is algebraic over $\wp_\Omega(u)$, then $\xi' \in \Xi(\Omega,\xi)$ and, in particular, $K_\omega^*\xi \subset \Xi(\omega,\xi)$, for each $\omega \in \mathbb{C}\setminus\mathbb{R}$.*
(4) *$e^{qv}\widetilde{\sigma}_{\Omega,q\xi}(u)$ is algebraic over $\mathbb{C}\left(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)\right)$, for each $q \in \mathbb{Q}^*$.*
(5) *If $\xi' \in \Omega$, there exists $c \in \mathbb{C}$ such that $\left(\wp_\Omega(u), e^{cu+v}\widetilde{\sigma}_{\Omega,\xi+\xi'}(u)\right)$ is algebraic over $\mathbb{C}\left(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)\right)$. In particular, $\Omega+\xi \subset \Xi(\Omega,\xi)$.*

Proof. For (1), reflexivity and transitivity of $\sim_\Omega$ are clear from the definition. To show that $\sim_\Omega$ is symmetric, suppose that $\xi \sim_\Omega \xi'$, *i.e.*, there exists $\alpha \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u)) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$. Composing with α^{-1}, we infer that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))\circ\alpha^{-1}$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u))$. So $\xi' \sim_\Omega \xi$.

For (2), note that, by Lemma 4.20, for each $z \in \mathbb{C}$, there is $\alpha_z \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,z}(u)) \circ \alpha_z$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',z}(u))$. In particular, $(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',z}(u)) \circ \alpha_z^{-1}$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,z}(u))$. For each $\xi' \in \Xi(\Omega',\xi)$, there is $\alpha \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi'}(u))\circ\alpha$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u))$. Thus, $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u))\circ\alpha_{\xi'}\circ\alpha\circ\alpha_\xi^{-1}$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$, so $\xi' \in \Xi(\Omega,\xi)$. Similarly, for any given $\xi' \in \Xi(\Omega,\xi)$ there exists $\beta \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u)) \circ \beta$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$. Thus, $(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi'}(u))\circ\alpha_{\xi'}^{-1}\circ\beta\circ\alpha_\xi$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u))$, so $\xi' \in \Xi(\Omega',\xi)$.

For (3), let $\Omega' := \xi'^{-1}\xi\Omega$. By Fact 4.14, $\wp_{\Omega'}(u) = \xi'^2\xi^{-2}\wp_\Omega(\xi'\xi^{-1}u)$, so $\wp_{\Omega'}(u)$ is algebraic over $\mathbb{C}(\wp_\Omega(u))$. By Lemma 4.6, there exists a lattice $\Omega'' < (\mathbb{C},+)$ such that both $\Omega'' < \Omega$ and $\Omega'' < \Omega'$. By Lemma 4.20, there exist $\alpha,\beta \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_{\Omega''}(u), e^v\widetilde{\sigma}_{\Omega'',\xi}(u))$ and $(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u))\circ\beta$ is algebraic over $\mathbb{C}(\wp_{\Omega''}(u), e^v\widetilde{\sigma}_{\Omega'',\xi}(u))$. Let $\gamma(u,v) = (\xi'\xi^{-1}u, v)$. Applying Fact 4.14, one checks that $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u)) \circ \gamma$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u))$. Thus, $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi'}(u)) \circ \gamma \circ \beta \circ \alpha^{-1}$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$, so $\xi' \in \Xi(\Omega,\xi)$. The additional condition follows taking $\xi' = a\xi$ and taking into account that, by Proposition 4.8, $\wp_\Omega(au)$ is algebraic over $\mathbb{C}(\wp_\Omega(u))$ if and only if $a \in K_\omega^*$.

We will prove (4) in three steps.

Claim (1). *For each lattice Ω of $(\mathbb{C},+)$,*

$$\mathbb{C}\left(\wp_\Omega(u), e^{-v}\widetilde{\sigma}_{\Omega,-\xi}(u)\right) = \mathbb{C}\left(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u)\right).$$

Furthermore, $\Omega \subset \Xi(\Omega, 0)$.

Proof of Claim 1. We first consider the case where $\xi \notin \Omega$. By [**11**, Ch.IV, §3, Corollary],

$$\wp_\Omega(u) - \wp_\Omega(z) = -\frac{\sigma_\Omega(u+z)\sigma_\Omega(u-z)}{\sigma^2_\Omega(u)\sigma^2_\Omega(z)}.$$

Evaluating the above expression at $z = \xi$ and solving for $\frac{\sigma_\Omega(u+\xi)}{\sigma_\Omega(u)}$, we deduce that

$$e^{-v}\frac{\sigma_\Omega(u+\xi)}{\sigma_\Omega(u)} = (\wp_\Omega(u) - \wp_\Omega(\xi))\left(e^{-v}\frac{\sigma_\Omega(u)}{\sigma_\Omega(u-\xi)}\right)\sigma^2_\Omega(\xi).$$

The claim follows easily from this, so we are done with this case. Now, consider the case when $\xi \in \Omega$. Since $\wp_\Omega(u-\xi) = \wp_\Omega(u)$, there exists $C \in \mathbb{C}$ such that $\zeta_\Omega(u-\xi) = \zeta_\Omega(u) + C$, so there also exists $D \in \mathbb{C}$ such that $\widetilde{\sigma}_{\Omega,\xi}(u) = e^{Cu+D}$. Since σ_Ω is an odd function, we get that $\widetilde{\sigma}_{\Omega,-\xi}(u) = \widetilde{\sigma}_{\Omega,\xi}(-u) = e^{-Cu+D}$ and, therefore, $e^{-v}\widetilde{\sigma}_{\Omega,-\xi}(u) = e^{-v}e^{-Cu+D} = e^{2D}(e^{v}e^{Cu+D})^{-1} = e^{2D}(e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))^{-1}$, which shows the first part. For the remaining condition, let $\alpha(u,v) = (u, -Cu+v)$ and note that $(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u)) \circ \alpha = (\wp_\Omega(u), e^{v+D})$. Since $\alpha \in \mathrm{GL}_2(\mathbb{C})$, this shows that $\xi \in \Xi(\Omega, 0)$. Consequently, $\Omega \subset \Xi(\Omega, 0)$. □

Claim (2): *For each given lattice Ω of $(\mathbb{C},+)$ and for each $n \in \mathbb{N}^*$,*

$$(\wp_\Omega(u), e^{nv}\widetilde{\sigma}_{\Omega,n\xi}(u)) \text{ is algebraic over } \mathbb{C}\left(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u)\right).$$

Proof of Claim (2). We prove it by induction. The case $n = 1$ is trivial. Suppose that $e^{(n-1)v}\widetilde{\sigma}_{\Omega,(n-1)\xi}(u)$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))$. By Claim (1), $(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{-v}\widetilde{\sigma}_{\Omega,-\xi}(u))$. Thus, $e^{(n-1)v}\widetilde{\sigma}_{\Omega,(n-1)\xi}(u)$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{-v}\widetilde{\sigma}_{\Omega,-\xi}(u))$. Changing (u,v) by $(u-\xi, v)$, we get that $e^{(n-1)v}\widetilde{\sigma}_{\Omega,(n-1)\xi}(u-\xi)$ is algebraic over $\mathbb{C}(\wp_\Omega(u-\xi), e^{-v}(\widetilde{\sigma}_{\Omega,\xi}(u))^{-1})$. By Corollary 3.5, $\wp_\Omega(u-\xi)$ is algebraic over $\mathbb{C}(\wp_\Omega(u))$, so $e^{(n-1)v}\widetilde{\sigma}_{\Omega,(n-1)\xi}(u-\xi)$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))$. Since $e^{nv}\widetilde{\sigma}_{\Omega,n\xi}(u) = e^{(n-1)v}\widetilde{\sigma}_{\Omega,(n-1)\xi}(u-\xi)\cdot e^{v}\widetilde{\sigma}_{\Omega,\xi}(u)$, we conclude that $e^{nv}\widetilde{\sigma}_{\Omega,n\xi}(u)$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))$. □

Let us finish the proof of (4). Given $q \in \mathbb{Q}^*$, let $a, b \in \mathbb{Z}^*$ such that a/b is the irreducible fraction of q. By Claim (1) and (2), the couple $(\wp_\Omega(u), e^{nv}\widetilde{\sigma}_{\Omega,n\xi}(u))$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{v}\widetilde{\sigma}_{\Omega,\xi}(u))$, for each lattice Ω of $(\mathbb{C},+)$ and each $n \in \mathbb{Z}^*$. In particular, both $(\wp_{b\Omega}(u), e^{av}\widetilde{\sigma}_{b\Omega,a\xi}(u))$ and $(\wp_{b\Omega}(u), e^{bv}\widetilde{\sigma}_{b\Omega,b\xi}(u))$ are algebraic over $\mathbb{C}(\wp_{b\Omega}(u), e^{v}\widetilde{\sigma}_{b\Omega,\xi}(u))$. Therefore, $(\wp_{b\Omega}(u), e^{av}\widetilde{\sigma}_{b\Omega,a\xi}(u))$ is algebraic over $\mathbb{C}(\wp_{b\Omega}(u), e^{bv}\widetilde{\sigma}_{b\Omega,b\xi}(u))$. If we change u by bu and v by $b^{-1}v$, we get that $(\wp_{b\Omega}(bu), e^{qv}\widetilde{\sigma}_{b\Omega,a\xi}(bu))$ is algebraic over $\mathbb{C}(\wp_{b\Omega}(bu), e^{v}\widetilde{\sigma}_{b\Omega,b\xi}(bu))$. Now, we apply Fact 4.14.

Finally, we prove (5). First note that

$$\widetilde{\sigma}_{\Omega,\xi+\xi'}(u) = \frac{\sigma_\Omega(u-\xi-\xi')}{\sigma_\Omega(u-\xi')}\frac{\sigma_\Omega(u-\xi')}{\sigma_\Omega(u)} = \widetilde{\sigma}_{\Omega,\xi}(u-\xi')\widetilde{\sigma}_{\Omega,\xi'}(u).$$

Since $\xi' \in \Omega$, it follows from the proof of Claim (1) of Lemma 4.21.(4) that $\widetilde{\sigma}_{\Omega,\xi'}(u) = e^{Cu+D}$, for some $C, D \in \mathbb{C}$, and, therefore,

$$(\wp_\Omega(u), e^{-D-Cu+v}\widetilde{\sigma}_{\Omega,\xi+\xi'}(u)) = (\wp_\Omega(u-\xi'), e^v\widetilde{\sigma}_{\Omega,\xi}(u-\xi')).$$

Now, by Corollary 3.5, we deduce that

$$(\wp_\Omega(u-\xi'), e^v\widetilde{\sigma}_{\Omega,\xi}(u-\xi')) \text{ is algebraic over } \mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)).$$

Hence, taking $c := -C$ and noting that e^{-D} is a constant in $\mathbb{C}$, we conclude that

$$(\wp_\Omega(u), e^{cu+v}\widetilde{\sigma}_{\Omega,\xi+\xi'}(u)) \text{ is algebraic over } \mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)),$$

as requested. □

The next lemma is the counterpart to claim inside Lemma 4.18 for groups of the eighth type.

LEMMA 4.22. *Let Ω be a lattice of $(\mathbb{C},+)$, let $\xi_1, \xi_2 \in \mathbb{C}$ and let $\alpha(u,v) := (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$. If $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi_2}(u)) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi_1}(u))$, then $a \neq 0$, $\wp_\Omega(au)$ is algebraic over $\wp_\Omega(u)$, $b = 0$, $d \in \mathbb{Q}^*$ and $a^{-1}\xi_2 - q\xi_1 \in \Omega$ for certain $q \in \mathbb{Q}^*$. In addition, if $a = 1$ then $\xi_2 - d\xi_1 \in \Omega$.*

PROOF. Suppose that $b \neq 0$. Assume first, for a contradiction, that also $a \neq 0$. We recall that $\wp_{\Omega_2}$ admits an AAT, so $\wp_{\Omega_2}(au+bv)$ is algebraic over $\mathbb{C}(\wp_{\Omega_2}(au), \wp_{\Omega_2}(bv))$. Hence, $\wp_{\Omega_2}(bv)$ is algebraic over $\mathbb{C}(\wp_{\Omega_2}(au+bv), \wp_{\Omega_2}(au))$. So $\wp_{\Omega_2}(bv)$ is also algebraic over

$$\mathbb{C}\left(\wp_{\Omega_2}(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega_2,\xi_2}(au+bv), \wp_{\Omega_2}(au)\right).$$

By hypothesis, we also have that

$$\left(\wp_{\Omega_2}(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega_2,\xi_2}(au+bv)\right)$$

is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$, so $\wp_{\Omega_2}(bv)$ is algebraic over

$$\mathbb{C}\left(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u), \wp_{\Omega_2}(au)\right).$$

Since $\wp_{\Omega_2}(bv)$ only depends on v, we get that $\wp_{\Omega_2}(bv)$ is algebraic over $\mathbb{C}(e^v)$. This contradicts Fact 4.17. If $a = 0$ then we achieve a similar contradiction, because $\wp_{\Omega_2}(bv)$ would be algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$. Both contradictions imply $b = 0$ and, since $\alpha \in GL_2(\mathbb{C})$, that $a \neq 0$.

Since $b = 0$, we get that $\wp_{\Omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$. If $\wp_{\Omega_2}(au)$ is not algebraic over $\mathbb{C}(\wp_{\Omega_1}(u))$ then $e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), \wp_{\Omega_2}(au))$, which is impossible because the elements of this field does not depend on the variable v.

Now, note that $e^{cu+dv}\widetilde{\sigma}_{\Omega_2,\xi_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$. Thus, if we evaluate u in a point that does not belong to $a^{-1}\Omega_2 \cup \Omega_1$, we conclude that e^{dv} is algebraic over $\mathbb{C}(e^v)$, so $d \in \mathbb{Q}$. Since $b = 0$, we have that $d \in \mathbb{Q}^*$, as required.

We have proved that $\wp_\Omega(au)$ is algebraic over $\wp_\Omega(u)$, $b = 0$ and $d \in \mathbb{Q}^*$. Thus, by Fact 4.14, we have that $\wp_{a^{-1}\Omega}(u) \in \mathbb{C}(\wp_\Omega(au))$, so it is algebraic over $\mathbb{C}(\wp_\Omega(u))$. Thus, by Lemma 4.6, there exists a lattice $\Omega' < (\mathbb{C},+)$ such that $\Omega' < \Omega$ and $\Omega' < a^{-1}\Omega$.

By hypothesis and by Fact 4.14, $(\wp_{a^{-1}\Omega}(u), e^{cu+dv}\widetilde{\sigma}_{a^{-1}\Omega,a^{-1}\xi_2}(u))$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi_1}(u))$. Our aim now is to derive from the latter an algebraic relation that is uniform with respect to Ω'.

On one hand, defining $\beta_1(u,v) := (u, a^{-1}\xi_2\mathfrak{c}(a^{-1}\Omega,\Omega')u + [a^{-1}\Omega : \Omega']v)$, then, by Lemma 4.20, $(\wp_{a^{-1}\Omega}(u), e^{cu+dv}\widetilde{\sigma}_{a^{-1}\Omega,a^{-1}\xi_2}(u)) \circ \beta_1$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^{cu+dv}\widetilde{\sigma}_{\Omega',a^{-1}\xi_2}(u))$. On the other hand, if we define $\beta_2(u,v) := (u, \xi_1\mathfrak{c}(\Omega,\Omega')u + [\Omega : \Omega']v)$ then, by Lemma 4.20, $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi_1}(u)) \circ \beta_2$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi_1}(u))$. All in all, we deduce that

$$(\wp_{\Omega'}(u), e^{cu+dv}\widetilde{\sigma}_{\Omega',a^{-1}\xi_2}(u)) \circ \beta_1^{-1} \circ \beta_2$$

is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi_1}(u))$. Making some computations, we deduce that

$$e^{c'u+d'v}\widetilde{\sigma}_{\Omega',a^{-1}\xi_2}(u) \text{ is algebraic over } \mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi_1}(u)),$$

where

$$c' := c + d\frac{\xi_1\mathfrak{c}(\Omega,\Omega') - a^{-1}\xi_2\mathfrak{c}(a^{-1}\Omega,\Omega')}{[a^{-1}\Omega : \Omega']} \quad \text{and} \quad d' := d\frac{[\Omega : \Omega']}{[a^{-1}\Omega : \Omega']} \in \mathbb{Q}^*.$$

Next, by Lemma 4.21.(4) and since $d' \in \mathbb{Q}^*$, we get that $e^{-d'v}\widetilde{\sigma}_{\Omega',-d'\xi_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi_1}(u))$. Thus, taking the product by the above function, we deduce that $e^{c'u}\widetilde{\sigma}_{\Omega',a^{-1}\xi_2}(au)\widetilde{\sigma}_{\Omega',-d'\xi_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi_1}(u))$ and, in particular, over $\mathbb{C}(\wp_{\Omega'}(u))$.

By [**11**, Ch.IV, Theorem 4], the function

$$\widetilde{\sigma}_{\Omega',a^{-1}\xi_2}(u) \cdot \widetilde{\sigma}_{\Omega',-d'\xi_1}(u) \cdot \widetilde{\sigma}_{\Omega',-a^{-1}\xi_2+d'\xi_1}(u)$$

is elliptic of period Ω', so it is algebraic over $\mathbb{C}(\wp_{\Omega'}(u))$. In particular, $f(u) := e^{-c'u}\widetilde{\sigma}_{\Omega',-a^{-1}\xi_2+d'\xi_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\Omega'}(u))$, which means that $f(u)$ is an elliptic function. As, it has a unique simple pole in its fundamental domain (see [**11**, Ch.IV, §2]), so $f(u)$ must be constant ([**11**, Ch.II, Corollary to Theorem 2]). Thus, there exists $C \in \mathbb{C}$ such that $\widetilde{\sigma}_{\Omega',-a^{-1}\xi_2+d'\xi_1}(u) = e^{C+c'u}$. Consequently,

$$\ln\left(\sigma_{\Omega'}(u - (-a^{-1}\xi_2 + d'\xi_1))\right) - \ln(\sigma_{\Omega'}(u)) = C + c'u.$$

Differentiating twice, we get

$$\wp_{\Omega'}(u - (-a^{-1}\xi_2 + d'\xi_1)) - \wp_{\Omega'}(u) = 0,$$

so $-a^{-1}\xi_2 + d'\xi_1 \in \Omega' < \Omega$, so it is enough to define $q := d'$, as required. Finally, note that in the particular case $a = 1$ we get that $d' = d$. $\square$

Lemmas 4.21 and 4.22 gives us an easy characterization of the sets $\Xi(\omega,\xi)$, for $\xi \in \mathbb{C}$ and $\omega \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 4.23. *Let $\omega \in \mathbb{C} \setminus \mathbb{R}$. Then, $\Xi(\omega,\xi) = \langle 1,\omega\rangle_{\mathbb{Q}} + K_\omega^*\xi$ for each $\xi \in \mathbb{C}$.*

Proof. Fix $\xi \in \mathbb{C}$ and denote $\Omega := \langle 1,\omega\rangle_{\mathbb{Z}}$. Firstly, let us show that $\Xi(\Omega,\xi) \subset \langle 1,\omega\rangle_{\mathbb{Q}} + K_\omega^*\xi$.

Indeed, take $\xi' \in \Xi(\Omega,\xi)$. By Lemma 4.22, there exist $a \in K_\omega^*$ and $q \in \mathbb{Q}^*$ such that $a^{-1}\xi' - q\xi \in \Omega \subset \langle 1,\omega\rangle_{\mathbb{Q}}$. Recall that either $K_\omega = \mathbb{Q}(\omega) = \langle 1,\omega\rangle_{\mathbb{Q}}$ if ω is quadratic over $\mathbb{Q}$ or $K_\omega = \mathbb{Q}$ otherwise. In any case, K_ω^* is contained in $\langle 1,\omega\rangle_{\mathbb{Q}}^*$ and, therefore, $\xi' \in \langle 1,\omega\rangle_{\mathbb{Q}} + K_\omega^*\xi$, as required.

For the other inclusion, let us see first the following:

Claim. $\lambda+\xi\in\Xi(\omega,\xi)$, *for all* $\lambda\in\langle 1,\omega\rangle_{\mathbb{Q}}$.

Proof of the claim. There exists $n\in\mathbb{N}^*$ such that $n\lambda\in\langle 1,\omega\rangle_{\mathbb{Z}}$. In particular, $\lambda\in n^{-1}\Omega$. By Lemma 4.21.(5), $\lambda+\xi\in\Xi(n^{-1}\Omega,\xi)$. Since $\Omega<n^{-1}\Omega$, by Lemma 4.21.(2), we get $\lambda+\xi\in\Xi(\Omega,\xi)$. □

Let $\lambda\in\langle 1,\omega\rangle_{\mathbb{Q}}$ and $q\in K_\omega^*$, we show now that $\lambda+q\xi\in\Xi(\omega,\xi)$. Since $q\in K_\omega^*$, there exists by Fact 4.14 a lattice Ω' that is both a sublattice of Ω and $q^{-1}\Omega$. Then, by Lemma 4.20, there exists $\alpha_1\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi}(u)\big)\circ\alpha_1 \text{ is algebraic over } \mathbb{C}\big(\wp_{\Omega'}(u),e^v\widetilde{\sigma}_{\Omega',\xi}(u)\big).$$

Similarly, there exists $\alpha_2\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\big(\wp_{q^{-1}\Omega}(u),e^v\widetilde{\sigma}_{q^{-1}\Omega,\xi}(u)\big)\circ\alpha_2 \text{ is algebraic over } \mathbb{C}\big(\wp_{\Omega'}(u),e^v\widetilde{\sigma}_{\Omega',\xi}(u)\big).$$

Let $\beta_2:=\alpha_2\circ\alpha_1^{-1}$. In particular, $\big(\wp_{q^{-1}\Omega}(u),e^v\widetilde{\sigma}_{q^{-1}\Omega,\xi}(u)\big)\circ\beta_2$ is algebraic over $\mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi}(u)\big)$. Moreover, if we define $\alpha_3(u,v):=(qu,v)$ and $\beta_3(u,v):=\alpha_3\circ\beta_2$, then

$$\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,q\xi}(u)\big)\circ\beta_3 \text{ is algebraic over } \mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi}(u)\big).$$

By the claim above, there exists $\alpha_4\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\lambda+q\xi}(u)\big)\circ\alpha_4 \text{ is algebraic over } \mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,q\xi}(u)\big).$$

Thus, if we denote $\beta_4(u,v):=\alpha_4\circ\beta_3$, then $\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,q\xi+\lambda}(u)\big)\circ\beta_4$ is algebraic over $\mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi}(u)\big)$, as required. □

Finally, we study all the possible isomorphism classes between groups of the eighth type, in order to conclude this case.

LEMMA 4.24. *Fix* $\omega_1,\omega_2\in\mathbb{C}\setminus\mathbb{R}$, $\xi_1,\xi_2\in\mathbb{C}$. *There exist* $a,b,c,d\in\mathbb{Z}$ *such that* $ad-bc\neq 0$, $\omega_2=\frac{a\omega_1+b}{c\omega_1+d}$ *and* $(c\omega_1+d)\xi_2\in\Xi(\omega_1,\xi_1)$ *if and only if there exists* $\alpha\in\mathrm{GL}_2(\mathbb{C})$ *such that* $\big(\wp_{\omega_2}(u),e^v\widetilde{\sigma}_{\omega_2,\xi_2}(u)\big)\circ\alpha$ *is algebraic over* $\mathbb{C}\big(\wp_{\Omega_1}(u),e^v\widetilde{\sigma}_{\omega_1,\xi_1}(u)\big)$.

PROOF. Let us denote $\Omega_1:=\langle 1,\omega_1\rangle_{\mathbb{Z}}$ and $\Omega_2:=\langle 1,\omega_2\rangle_{\mathbb{Z}}$. We begin with the left to right implication. Let $\Omega:=\langle c\omega_1+d,a\omega_1+b\rangle_{\mathbb{Z}}$ and $\alpha_1(u,v):=((c\omega_1+d)u,v)$. By Lemma 4.19, we get that $\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,(c\omega_1+d)\xi_2}(u)\big)\circ\alpha_1$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_2}(u),e^v\widetilde{\sigma}_{\Omega_2,\xi_2}(u)\big)$. We note that $\Omega<\Omega_1$. Let $\alpha_2(u,v):=(u,\xi_1\mathfrak{c}(\Omega_1,\Omega)u+[\Omega_1:\Omega]v)$. By Lemma 4.20, $\big(\wp_{\Omega_1}(u),e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u)\big)\circ\alpha_2$ is algebraic over $\mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi_1}(u)\big)$. By Lemma 4.21.(2), $(c\omega_1+d)\xi_2\in\Xi(\Omega,\xi_1)$. So there exists $\alpha_3\in\mathrm{GL}_2(\mathbb{C})$, such that $\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,(c\omega_1+d)\xi_2}(u)\big)\circ\alpha_3$ is algebraic over $\mathbb{C}\big(\wp_\Omega(u),e^v\widetilde{\sigma}_{\Omega,\xi_1}(u)\big)$. So $\big(\wp_{\Omega_1}(u),e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u)\big)\circ\alpha_2\circ\alpha_3^{-1}\circ\alpha_1$ is algebraic over $\mathbb{C}\big(\wp_{\Omega_2}(u),e^v\widetilde{\sigma}_{\Omega_2,\xi_2}(u)\big)$. Define $\alpha:=\alpha_1^{-1}\circ\alpha_3\circ\alpha_2^{-1}$ and we are done.

Now, we prove the right to left implication. By hypothesis and Lemma 4.22, we have $\alpha(u,v)=(a'u,c'u+d'v)$, for some $a',c',d'\in\mathbb{C}$ such that $\wp_{\Omega_2}(a'u)$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u))$. Thus, by Lemma 3.9 and Theorem 4.7.(II), there exist $a,b,c,d\in\mathbb{Z}$ with $ad-bc\neq 0$ such that $\omega_2=\frac{a\omega_1+b}{c\omega_1+d}$ and $a'=\frac{n}{c\omega_1+d}$, for some $n\in\mathbb{N}^*$.

On the other hand, by Fact 4.14 and Lemma 4.6, there exists a lattice Ω of $(\mathbb{C},+)$ such that both $\Omega<\Omega_1$ and $\Omega<a'^{-1}\Omega_2$. By hypothesis,

$(\wp_{\Omega_2}(a'u), e^{c'u+d'v}\widetilde{\sigma}_{\Omega_2,\xi_2}(a'u))$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$, so, also by Fact 4.14, we have that $(\wp_{a'^{-1}\Omega_2}(u), e^{c'u+d'v}\widetilde{\sigma}_{a'^{-1}\Omega_2,a'^{-1}\xi_2}(u))$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u))$. By Lemma 4.20, there exists $\alpha_1 \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_{\Omega_1}(u), e^v\widetilde{\sigma}_{\Omega_1,\xi_1}(u)) \circ \alpha_1$ is algebraic over $\mathbb{C}(\wp_{\Omega}(u), e^v\widetilde{\sigma}_{\Omega,\xi_1}(u))$. Similarly, pick $\alpha_2 \in \mathrm{GL}_2(\mathbb{C})$ such that $(\wp_{a'^{-1}\Omega_2}(u), e^{c'u+d'v}\widetilde{\sigma}_{a'^{-1}\Omega_2,a'^{-1}\xi_2}(u))\circ \alpha_2$ is algebraic over $\mathbb{C}(\wp_{\Omega}(u), e^{c'u+d'v}\widetilde{\sigma}_{\Omega,a'^{-1}\xi_2}(u))$. Then, we deduce that $(\wp_{\Omega}(u), e^{c'u+d'v}\widetilde{\sigma}_{\Omega,a'^{-1}\xi_2}(u))\circ\alpha_2^{-1}\circ\alpha_1$ is algebraic over $\mathbb{C}(\wp_{\Omega}(u), e^v\widetilde{\sigma}_{\Omega,\xi_1}(u))$, so $a'^{-1}\xi_2 \in \varXi(\Omega, \xi_1)$. Thus, by Lemma 4.21.(1) and 4.21.(2), we have that

$$\varXi(\omega_1, \xi_1) = \varXi(\Omega, \xi_1) = \varXi(\Omega, a'^{-1}\xi_2) = \varXi(\omega_1, a'^{-1}\xi_2).$$

Finally, it follows from Lemma 4.21.(3) that

$$(cw + d)\xi_2 = na'^{-1}\xi_2 \in \varXi(\omega_1, a'^{-1}\xi_2) = \varXi(\omega_1, \xi_1),$$

as required. □

Corollary 4.25. *Let $\Omega = \langle 1, \omega\rangle_{\mathbb{Z}}$, for some $\omega \in \mathbb{C}\setminus\mathbb{R}$, and $\xi \in \mathbb{C}$. For each $q \in \mathbb{Z}^*$, $(\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u))\circ\alpha$ is algebraic over $\mathbb{C}(\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u))$, for $\alpha(u, v) = q^{-1}(u, -\xi\mathfrak{c}(\Omega, q\Omega)u + v)$.*

Proof. Fix $q \in \mathbb{Z}$. Let us apply Lemma 4.24 and its proof for $b = c = 0$ and $a = d = q$. In the proof of the left to right implication of Lemma 4.24 we obtain for this choice $\alpha_1(u, v) = (qu, v)$ and $\alpha_2(u, v) = (u, \xi\mathfrak{c}(\Omega, q\Omega)u + [\Omega : q\Omega]v)$. Also, by Lemma 4.21.(4), we can choose $\alpha_3(u, v) = (u, qv)$. Finally, since $[\Omega : q\Omega] = q^2$, we get that

$$\alpha(u, v) = \alpha_1^{-1}\circ\alpha_3\circ\alpha_2^{-1}(u, v) = q^{-1}(u, -\xi\mathfrak{c}(\Omega, q\Omega)u + v),$$

as required. □

We finish with some computations that will allow us to provide sharp expressions of the automorphisms. Recal that invariant $\mathfrak{c}$ was defined in Remark 4.4.

Definition-Proposition 4.26. *Let Ω_1 and Ω_2 be two lattices of $(\mathbb{C}, +)$ that have Ω as a common sublattice. We define*

$$[\Omega_2 : \Omega_1] := \frac{[\Omega_2 : \Omega]}{[\Omega_1 : \Omega]} \qquad \text{and} \qquad \mathfrak{qc}(\Omega_2, \Omega_1) := \mathfrak{c}(\Omega_2, \Omega) - \frac{[\Omega_2 : \Omega]}{[\Omega_1 : \Omega]}\mathfrak{c}(\Omega_1, \Omega).$$

The above definitions do not depend on the chosen sublattice Ω.

Proof. If we define $\alpha_1(u, v) := (u, \mathfrak{c}(\Omega_1, \Omega)u + [\Omega_1 : \Omega]v)$, $\alpha_2(u, v) := (u, \mathfrak{c}(\Omega_2, \Omega)u+[\Omega_2 : \Omega]v)$ and $\alpha := \alpha_2\circ\alpha_1^{-1}$ then, by Lemma 4.20, $(\wp_{\Omega_2}(u), v-\zeta_{\Omega_2}(u))\circ\alpha$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), v-\zeta_{\Omega_1}(u))$. Note that

$$\alpha(u, v) = \left(u, \left(\mathfrak{c}(\Omega_2, \Omega) - \frac{[\Omega_2 : \Omega]}{[\Omega_1 : \Omega]}\mathfrak{c}(\Omega_1, \Omega)\right)u + \frac{[\Omega_2 : \Omega]}{[\Omega_1 : \Omega]}v\right).$$

Suppose now that Ω' is another common sublattice of both Ω_1 and Ω_2. Then, $(\wp_{\Omega_2}(u), v-\zeta_{\Omega_2}(u))\circ\alpha'$ is algebraic over $\mathbb{C}(\wp_{\Omega_1}(u), v-\zeta_{\Omega_1}(u))$ for

$$\alpha'(u, v) = \left(u, \left(\mathfrak{c}(\Omega_2, \Omega') - \frac{[\Omega_2 : \Omega']}{[\Omega_1 : \Omega']}\mathfrak{c}(\Omega_1, \Omega')\right)u + \frac{[\Omega_2 : \Omega']}{[\Omega_1 : \Omega']}v\right).$$

In particular, $\alpha^{-1}\circ\alpha'$ is an automorphism of $(\mathbb{C}^2,+,(\wp_{\Omega_1}(u),v-\zeta_{\Omega_1}(u))$. To finish the proof, it is enough to show the following:

Claim. Let Ω be a lattice of $(\mathbb{C},+)$ and let $\beta(u,v):=(u,cu+dv)\in \mathrm{GL}_2(\mathbb{C})$ be an automorphism of $(\mathbb{C}^2,+,(\wp_\Omega(u),v-\zeta_\Omega(u))$. Then, β is the identity.

Proof of the claim. Indeed, $cu+dv-\zeta_\omega(u)$ is algebraic over $\mathbb{C}(\wp_\omega(u),v-\zeta_\omega(u))$. If we take $v=\zeta_\omega(u)$, we obtain that $f(u):=cu+(d-1)\zeta_\omega(u)$ is algebraic over $\mathbb{C}(\wp_\omega(u))$. In particular, $f(u)$ is an elliptic function. On the other hand, $f(u)$ has a unique simple pole in the fundamental period-parallelogram and, consequently, $f(u)$ is a constant (see [**11**, Ch.III, §4] and [**11**, Ch.II, Corollary to Theorem 2]). Thus, there exists $C\in\mathbb{C}$ such that $(d-1)\zeta_\omega(u)=C-cu$. It follows that $d=1$ and $c=0$, as required. □

LEMMA 4.27. *Let Ω and Ω' be lattices of $(\mathbb{C},+)$ and $a\in\mathbb{C}^*$. Then,*

(1) *$\mathfrak{qc}(\Omega,\Omega')=\mathfrak{c}(\Omega,\Omega')$ if Ω' is a sublattice of Ω.*
(2) *$\mathfrak{qc}(\Omega,\Omega')=a^2\mathfrak{qc}(a\Omega,a\Omega')$ if Ω and Ω' have a common sublattice.*
(3) *$\mathfrak{qc}(\Omega,\frac{q_1}{q_2}\Omega)=\mathfrak{c}(\Omega,q_1\Omega)-\mathfrak{c}(\Omega,q_2\Omega)$ for all $q_1,q_2\in\mathbb{Z}^*$.*

PROOF. For (1), it is enough to note that $\mathfrak{c}(\Omega',\Omega')=0$ and, therefore,

$$\mathfrak{qc}(\Omega,\Omega')=\mathfrak{c}(\Omega,\Omega')-\frac{[\Omega:\Omega']}{[\Omega':\Omega']}\mathfrak{c}(\Omega',\Omega')=\mathfrak{c}(\Omega,\Omega').$$

For (2), let Ω'' be a common sublattice of Ω and Ω'. Take $b_1:=0,b_2,\dots,b_n$ representatives of the cosets of the quotient of Ω by Ω''. Then, from the definition of $\mathfrak{c}$, it follows that $\mathfrak{c}(\Omega,\Omega'')=\sum_{i=2}^n\wp_{\Omega'}(b_i)$. Clearly, $ab_1=0,ab_2,\dots,ab_n$ are representatives of the cosets of the quotient of $a\Omega$ by $a\Omega''$ so that, by Fact 4.14, we obtain

$$\mathfrak{c}(a\Omega,a\Omega'')=\sum_{i=2}^n\wp_{a\Omega''}(ab_i)=a^{-2}\sum_{i=2}^n\wp_{\Omega''}(b_i)=a^{-2}\mathfrak{c}(\Omega,\Omega'').$$

Similarly, $\mathfrak{c}(a\Omega',a\Omega'')=a^{-2}\mathfrak{c}(\Omega',\Omega'')$. Therefore,

$$\mathfrak{qc}(a\Omega,a\Omega')=\mathfrak{c}(a\Omega,a\Omega'')-\frac{[a\Omega:a\Omega'']}{[a\Omega':a\Omega'']}\mathfrak{c}(a\Omega',a\Omega'')=a^{-2}\mathfrak{qc}(\Omega,\Omega').$$

Finally, for (3), just note that $q_1\Omega$ is a common sublattice of Ω and $\frac{q_1}{q_2}\Omega$. By definition and properties (1) and (2), we deduce

$$\begin{aligned}\mathfrak{qc}(\Omega,\tfrac{q_1}{q_2}\Omega)=&\mathfrak{c}(\Omega,q_1\Omega)-\frac{[\Omega:q_1\Omega]}{[\frac{q_1}{q_2}\Omega:q_1\Omega]}\mathfrak{c}(\tfrac{q_1}{q_2}\Omega,q_1\Omega)=\\=&\mathfrak{c}(\Omega,q_1\Omega)-\tfrac{q_1^2}{q_2^2}(\tfrac{q_1}{q_2})^{-2}\mathfrak{c}(\Omega,q_2\Omega),\end{aligned}$$

as required. □

2.3. Proof of the main results.

THEOREM 4.28. *(I) Every two-dimensional simply connected abelian locally $\mathbb{C}$-Nash group is isomorphic to a group of one and only one of the following types:*

- *Direct products of one-dimensional locally $\mathbb{C}$-Nash groups:*
  (1) $(\mathbb{C}^2,+,\mathrm{id}\times\mathrm{id})$.

(2) $(\mathbb{C}^2, +, \exp \times \mathrm{id})$.
(3) $(\mathbb{C}^2, +, \wp_\omega \times \mathrm{id})$, *for some* $\omega \in \mathbb{C} \setminus \mathbb{R}$.
(4) $(\mathbb{C}^2, +, \exp \times \exp)$.
(5) $(\mathbb{C}^2, +, \wp_\omega \times \exp)$, *for some* $\omega \in \mathbb{C} \setminus \mathbb{R}$.
(6) $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$, *for some* $\omega_1, \omega_2 \in \mathbb{C} \setminus \mathbb{R}$.

- *Not direct products of one-dimensional locally* $\mathbb{C}$*-Nash groups:*

(7) $(\mathbb{C}^2, +, (\wp_\omega(u), v - \zeta_\omega(u))$, *for some* $\omega \in \mathbb{C} \setminus \mathbb{R}$.
(8) $(\mathbb{C}^2, +, (\wp_\omega(u), e^v \widetilde{\sigma}_{\omega,\xi}(u)))$, *for some* $\omega \in \mathbb{C} \setminus \mathbb{R}$ *and* $\xi \notin \langle 1, \omega \rangle_{\mathbb{Q}}$.
(9) *Universal covering groups of complex abelian surfaces which are not direct products of elliptic curves.*

(II) *The isomorphism classes of groups of types* (1) *to* (9) *are defined as follows:*

(i) *Two direct products are isomorphic if and only if their factor groups are isomorphic.*
(ii) *Two groups of the seventh type defined by data* ω_1 *and* ω_2*, respectively, are isomorphic if and only if there exist* $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc \neq 0$ *such that* $\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}$.
(iii) *Two groups of the eighth type defined by data* (ω_1, ξ_1) *and* (ω_2, ξ_2)*, respectively, are isomorphic if and only if there exist* $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc \neq 0$ *such that* $\omega_2 = \frac{a\omega_1 + b}{c\omega_1 + d}$ *and* $(c\omega_1 + d)\xi_2 \in \langle 1, \omega_1 \rangle_{\mathbb{Q}} + \xi_1 K^*_{\omega_1}$*, where* K_{ω_1} *is either* $\mathbb{Q}(\omega_1)$ *if* ω_1 *is quadratic over* $\mathbb{Q}$ *or* $\mathbb{Q}$ *otherwise.*
(iv) *Two groups that both are universal covering groups of abelian surfaces — groups of type sixth and ninth — are isomorphic if and only if the corresponding abelian surfaces are isogenous as abelian varieties.*

Proof. (I) By Theorem 3.8, any two-dimensional simply connected abelian locally $\mathbb{C}$-Nash group is isomorphic to $(\mathbb{C}^n, +, f)$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is a meromorphic map admitting an AAT. By *Painlevé's description* (see Subsection 2.1, specially Fact 4.10) and by Lemma 3.9, we can assume that f is exactly one of the functions defining the first five families of the Painlevé's description or that f is algebraic over a field of the six family $\mathcal{P}_6$. Moreover, by Lemmas 4.15 and 4.19, we can assume that all the lattices of $(\mathbb{C}, +)$ are of the form $\langle 1, \omega \rangle_{\mathbb{Z}}$ for some $\omega \in \mathbb{C} \setminus \mathbb{R}$. Thus, depending to which family belong the coordinate functions of f, we obtain the following cases:

- If they are in $\mathcal{P}_1$, we obtain a group of the first type.
- If they are in $\mathcal{P}_2$, we obtain a group of the second type.
- If they are in $\mathcal{P}_3$, we obtain a group of the fourth type.
- If they are in $\mathcal{P}_4$, we obtain a group of the third or seventh type.
- If they are in $\mathcal{P}_5$, we obtain a group of the fifth or eight type.
- If they are in $\mathcal{P}_6$, we obtain a group of the sixth or ninth type.

To finish the proof of (I), it is enough to show that if the coordinate functions of f are in $\mathcal{P}_6$ then $(\mathbb{C}^2, +, f)$ is isomorphic to the universal covering of an abelian surface.

Suppose that the coordinates functions of $f := (f_1, f_2)$ are algebraic over a field of the family $\mathcal{P}_6$, *i.e.*, over a field of the form $\mathbb{C}(\Lambda)$ for some lattice $\Lambda < (\mathbb{C}^2, +)$ and satisfying $\mathrm{tr.deg.}_{\mathbb{C}}\, \mathbb{C}(\Lambda) = 2$. Recall that $\mathbb{C}(\Lambda)$

is by definition the field of meromorphic functions $g : \mathbb{C}^2 \dashrightarrow \mathbb{C}$ satisfying $g(z) = g(z+\lambda)$ for all $\lambda \in \Lambda$. It is well-known that $\mathbb{C}^2/\Lambda$ is an abelian variety. Thus, there exists an analytic universal covering $(\mathbb{C}^2, +, f) \to \mathbb{C}^2/\Lambda$. However, we have to take care that the above covering lies properly in the locally $\mathbb{C}$-Nash category. For that reason, we must describe the algebraic structure of $\mathbb{C}^2/\Lambda$.

By [**45**, Ch.V, §12, Theorem 1], there are theta functions $g_0(z), \ldots, g_m(z)$ whose quotients generate $\mathbb{C}(\Lambda)$ and such that if we write out all the homogeneous algebraic relations of the form $P(g_0(z), \ldots, g_m(z)) = 0$ then the equations $P(x_0, \ldots, x_m) = 0$ define a nonsingular irreducible algebraic variety $\mathcal{N}$ in a m-dimensional projective space, which may be mapped biregularly onto the period torus $\mathcal{T} := \mathbb{C}^2/\Lambda$ by the correspondence $x_i = g_i(z)$ for $i \in \{0, \ldots, m\}$. Thus, define

$$\Phi : \mathcal{T} \to \mathcal{N} : z \mapsto (g_0(z) : \ldots : g_m(z)).$$

We can assume that $g_0(0) \neq 0$. Clearly, the addition group operation in $\mathcal{T}$ induces a group operation in $\mathcal{N}$ via

$$a \oplus b := \Phi(\Phi^{-1}(a) + \Phi^{-1}(b)), \quad a, b \in \mathcal{N}.$$

This group operation is regular and $\mathcal{N}$ is an algebraic group (see the paragraph above [**45**, Ch.V, §13, Theorem 2]). Now, consider the continuous homomorphism

$$\Psi : (\mathbb{C}^2, +, f) \to \mathcal{N} : (u, v) \mapsto \Phi(\bar{u}, \bar{v})$$

and let us show that it is a locally $\mathbb{C}$-Nash map. We can assume that a chart of the identity of the locally $\mathbb{C}$-Nash structure on $\mathcal{N}$ is given by the restriction of the projection $\pi : \mathcal{N} \to \mathbb{C}^2 : (x_0 : \ldots : x_m) \mapsto (x_1 x_0^{-1}, x_2 x_0^{-1})$ to some open neighborhood W of the identity of $\mathcal{N}$. By Proposition 2.9, the map Ψ is a locally $\mathbb{C}$-Nash map if $\pi \circ \Psi \circ f^{-1}$ is a $\mathbb{C}$-Nash map. In other words, if $(g_1 g_0^{-1}, g_2 g_0^{-1})(u, v)$ is algebraic over $f(u, v)$. Recall that the theta functions satisfy

$$\mathbb{C}(\Lambda) = \mathbb{C}(g_i g_j^{-1} \mid 1 \leq i, j \leq m) = \mathbb{C}(g_i g_0^{-1} \mid 1 \leq i \leq m).$$

Since $\operatorname{tr.deg.}_{\mathbb{C}} \mathbb{C}(\Lambda) = 2$ and we are assuming that $g_1 g_0^{-1}$ and $g_2 g_0^{-1}$ are algebraically independent, it follows that $\mathbb{C}(\Lambda)$ is an algebraic extension of $\mathbb{C}(g_1 g_0^{-1}, g_2 g_0^{-1})$. In particular, f is algebraic over $\mathbb{C}(g_1 g_0^{-1}, g_2 g_0^{-1})$. This shows that $(g_1 g_0^{-1}, g_2 g_0^{-1})(u, v)$ is algebraic over $f(u, v)$, as required.

(II) We first show that groups originated from different Painlevé's families cannot be isomorphic. Consider $(\mathbb{C}^2, +, f)$ and $(\mathbb{C}^2, +, g)$ for some meromorphic maps $f, g : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ that admit an AAT and such that f is algebraic over $\mathbb{L} \in \mathcal{P}_i$ and g is algebraic over $\mathbb{L}' \in \mathcal{P}_j$, for some $i, j \in \{1, \ldots, 6\}$. Let us show that if $(\mathbb{C}^2, +, f)$ and $(\mathbb{C}^2, +, g)$ are isomorphic then $i = j$.

Indeed, by Proposition 3.17, $r := \operatorname{rank} \Lambda_g = \operatorname{rank} \Lambda_f$. By Proposition 4.13, some of the cases are already solved, namely: if $r = 0$ then $i = j = 1$; if $r = 1$ then $i = j = 2$; if $r = 3$ then $i = j = 5$; and if $r = 4$ then $i = j = 6$. For the case $r = 2$, suppose by contradiction that $i = 4$ and $j = 3$. By definition, there exist $\alpha_1, \alpha_2 \in \mathrm{GL}_2(\mathbb{C})$, $\xi \in \mathbb{C}$ and a lattice Ω of $(\mathbb{C}, +)$ such that f is algebraic over $\mathbb{L}_1 = \mathbb{C}(g_{4,\xi,\Omega} \circ \alpha_1)$ and g is algebraic over $\mathbb{L}_2 = \mathbb{C}(g_3 \circ \alpha_2)$. By Lemma 3.9, there exists $\alpha \in \mathrm{GL}_2(\mathbb{C})$ such that $g \circ \alpha$ is algebraic over $\mathbb{C}(f)$. Since the coordinate functions of $g \circ \alpha$ are algebraically independent over $\mathbb{C}$, we get that $g_3 \circ \alpha_3$ is algebraic over $\mathbb{C}(g_{4,\xi,\Omega})$, for

$$\alpha_3 := \alpha_2 \circ \alpha \circ \alpha_1^{-1} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

Since $g_{4,\xi,\Omega}$ is algebraic over $\mathbb{C}(\wp_\Omega(u), \zeta_\Omega(u), v)$, we have that

$$\left(e^{au+bv}, e^{cu+dv}\right) \text{ is algebraic over } \mathbb{C}(\wp_\Omega(u), \zeta_\Omega(u), v).$$

Since $\alpha_3 \in \mathrm{GL}_2(\mathbb{C})$, either $b \neq 0$ or $d \neq 0$. Without loss of generality, we may assume $b \neq 0$. We also note that e^{bv} is algebraic over $\mathbb{C}(e^{au}, e^{au+bv})$, so

$$e^{bv} \text{ is algebraic over } \mathbb{C}(e^{au}, \wp_\Omega(u), \zeta_\Omega(u), v),$$

which means that e^{bv} is algebraic over $\mathbb{C}(v)$, which is a contradiction. So either $i = j = 3$ or either $i = j = 4$.

Now, we show that there does not exist isomorphisms between groups of the third and the seventh types, between groups of the fifth and eighth types and between groups of the sixth and ninth types, respectively. The case of groups of sixth and ninth type is clear. We consider groups of the third and seventh type. Suppose that there exists an isomorphism

$$\alpha(u,v) : (\mathbb{C}^2, +, (\wp_{\omega_1}(u), v)) \to (\mathbb{C}^2, +, (\wp_{\omega_2}(u), v - \zeta_{\omega_2}(u))),$$

that will be denoted $\alpha(u,v) = (au + bv, cu + dv)$. Then, by Lemma 3.9, $(\wp_{\omega_2}(u), v - \zeta_{\omega_2}(u)) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), v)$. By the claim in the proof of Lemma 4.18 (for $\xi = 0$), we deduce that $b = 0$ and $\wp_{\omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$. So, $cu + dv - \zeta_{\omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\omega_2}(au), v)$. Then, evaluating $v = 0$, we get that $cu - \zeta_{\omega_2}(au)$ is algebraic over $\mathbb{C}(\wp_{\omega_2}(au))$, which contradicts Fact 4.17. So these groups are not isomorphic. Next, suppose that there exists an isomorphism

$$\alpha(u,v) : (\mathbb{C}^2, +, (\wp_{\omega_1}(u), e^v \widetilde{\sigma}_{\omega_1,\xi_1}(u)) \to (\mathbb{C}^2, +, (\wp_{\omega_2}(u), e^v)),$$

that we denote again $\alpha(u,v) = (au + bv, cu + dv)$. Then, by Lemma 3.9, $(\wp_{\omega_2}(u), e^v) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), e^v \widetilde{\sigma}_{\omega_1,\xi_1}(u))$, so by Lemma 4.24, $0 \in \Xi(\omega_1, \xi_1)$. By Lemma 4.21.(1), $\xi_1 \in \Xi(\omega_1, 0)$ and so, by Lemma 4.23, $\xi_1 \in \langle 1, \omega \rangle_{\mathbb{Q}}$. This is a contradiction, because groups of type eight satisfy $\xi_1 \notin \langle 1, \omega_1 \rangle_{\mathbb{Q}}$, as required.

Once we know that direct products of groups of different types are not isomorphic, the proof of (i) reduces to study direct products of the same type. This is straightforward and we write down the case of groups of type (6) as an example. Let $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ and $(\mathbb{C}^2, +, \wp_{\omega_3} \times \wp_{\omega_4})$ be groups of type (6) and let $\alpha : (\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2}) \to (\mathbb{C}^2, +, \wp_{\omega_3} \times \wp_{\omega_4})$ be an isomorphism, say $\alpha(u,v) = (au + bv, cu + dv)$. Then, $\wp_{\omega_3}(au + bv)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), \wp_{\omega_2}(v))$. If $\wp_{\omega_3}(au+bv)$ is algebraic over $\mathbb{C}(\wp_{\omega_2}(v))$ then $a = 0$ and $(\mathbb{C}, +, \wp_{\omega_3})$ and $(\mathbb{C}, +, \wp_{\omega_2})$ are isomorphic. If $\wp_{\omega_3}(au + bv)$ is not algebraic over $\mathbb{C}(\wp_{\omega_2}(v))$ then $\wp_{\omega_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\omega_3}(au + bv), \wp_{\omega_2}(v))$ and, in particular, it is algebraic over $\mathbb{C}(\wp_{\omega_3}(au), \wp_{\omega_3}(bv), \wp_{\omega_2}(v))$. We deduce that $\wp_{\omega_1}(u)$ is algebraic over $\mathbb{C}(\wp_{\omega_3}(au))$, so $(\mathbb{C}, +, \wp_{\omega_3})$ and $(\mathbb{C}, +, \wp_{\omega_1})$ are isomorphic. Similarly, we get that $(\mathbb{C}, +, \wp_{\omega_4})$ is isomorphic to $(\mathbb{C}, +, \wp_{\omega_1})$ or

$(\mathbb{C},+,\wp_{\omega_2})$. Moreover, in case that $(\mathbb{C},+,\wp_{w_1})$ and $(\mathbb{C},+,\wp_{w_2})$ are not isomorphic, repeating the argument with α^{-1} we conclude that both $(\mathbb{C},+,\wp_{\omega_3})$ and $(\mathbb{C},+,\wp_{\omega_4})$ are not isomorphic to the same factor.

Finally, the isomorphism classes of groups of the seventh type follow from Lemma 4.18. The isomorphism classes of groups of the eighth type follows from Lemmas 4.23 and 4.24. The statement concerning groups of ninth type follows from Corollary 2.16. □

As a consequence of Theorem 4.28, we prove next the following relation between two-dimensional simply connected abelian locally $\mathbb{C}$-Nash groups and algebraic groups (recall that algebraic groups have a natural structure of locally $\mathbb{C}$-Nash group by Corollary 2.15). In fact, note that this gives us another characterization of the two-dimensional simply connected abelian locally $\mathbb{C}$-Nash groups.

Corollary 4.29. *Each group from* (1) *to* (9) *is locally $\mathbb{C}$-Nash isomorphic to the universal covering of an algebraic group. For groups of type* (1) *to* (6)*, this algebraic group is a direct product of two of the following ones:* $(\mathbb{C},+)$, $(\mathbb{C}^*,\cdot)$ *or an elliptic curve. For groups of type* (7) *to* (9)*, we get an (specific) extension of an elliptic curve by* $(\mathbb{C},+)$*, an (specific) extension of an elliptic curve by* $(\mathbb{C}^*,\cdot)$ *and an abelian surface, respectively.*

Proof. We only need to study groups of the seventh and eighth type. We use extensively the reference [**17**], in which M. Hindry present explicit embeddings of two-dimensional algebraic groups in projective spaces (and whose existence was implicitly proved by J.-P. Serre). The notation used here is the one of that paper.

We begin with the case of groups of the seventh type, so we have a locally $\mathbb{C}$-Nash group of the form

$$(\mathbb{C}^2,+,(\wp_\omega(u),v-\zeta_\omega(u))),$$

for some $\omega\in\mathbb{C}\setminus\mathbb{R}$. We denote $\Omega_E:=\langle 1,\omega\rangle_{\mathbb{Z}}$ and define the elliptic curve $E=:\mathbb{C}/\Omega_E$. Recall that, since $\zeta_\omega'=-\wp_\omega$, for each $\lambda\in\Omega_E$, the difference $\zeta_\omega(u+\lambda)-\zeta_\omega(u)$ is a constant function that we denote $\eta(\lambda)$. Moreover, given $\lambda_1,\lambda_2\in\Omega_E$,

$$\zeta_\omega(u+\lambda_1+\lambda_2)=\zeta_\omega(u+\lambda_1)+\eta(\lambda_2)=\zeta_\omega(u)+\eta(\lambda_1)+\eta(\lambda_2)$$

and, therefore, $\eta(\lambda_1+\lambda_2)=\eta(\lambda_1)+\eta(\lambda_2)$.

Consider the map $\Phi:\mathbb{C}^2\to\mathbb{P}^5$, defined as

$$\begin{aligned}\Phi(u,v)=(1:\wp_\omega(u):\wp_\omega'(u):v-\zeta_\omega(u):\wp_\omega(u)(v-\zeta_\omega(u))-\frac{1}{2}\wp_\omega'(u)\\ :\wp_\omega'(u)(v-\zeta_\omega(u))-2\wp_\omega^2(u))\end{aligned}$$

if $u\notin\Omega$ and $\Phi(u,v)=(0:0:1:0:0:v-\eta(u))$ otherwise. By [**17**, pag 28, Théorème], Φ is a parametrization of a quasi-projective subvariety G of $\mathbb{P}^5$. The addition in $\mathbb{C}^2$ induces an algebraic group structure on G and the kernel of Φ is $\Omega_G:=\{(\lambda,\eta(\lambda))\mid\lambda\in\Omega_E\}$. Moreover, G is an extension of the elliptic curve E by $(\mathbb{C},+)$. Therefore, it is enough to show that Φ is a locally $\mathbb{C}$-Nash map when we equip $\mathbb{C}^2$ with the locally $\mathbb{C}$-Nash structure $(\mathbb{C}^2,+,(\wp_\omega(u),v-\zeta_\omega(u)))$.

We recall that $(\mathbb{C}^2, +, (\wp_\omega(u), v - \zeta_\omega(u)))$ is a notation that means that there exists $a \in \mathbb{C}$ such that $\phi_a(u,v) := (\wp_\omega(u+a), v - \zeta_\omega(u+a))$ is a chart of the identity of the locally $\mathbb{C}$-Nash group structure on $(\mathbb{C}^2, +)$ (see Remark 3.7). On the other hand, there is an open neighborhood W of the identity of G such that the projection $\pi : W \to \mathbb{C}^2 : (x_0 : \ldots : x_5) \mapsto (x_1/x_2, x_3/x_2)$ is a chart of the locally $\mathbb{C}$-Nash structure of G. By Proposition 2.9, it is enough to prove that $(\wp_\omega(u+a), v - \zeta_\omega(u+a))$ is algebraic over $\mathbb{C}(\psi(u,v))$, where $\psi(u,v) := (\wp_\omega(u)(\wp'_\omega(u))^{-1}, (v - \zeta_\omega(u))(\wp'_\omega(u))^{-1})$. In fact, by Corollary 3.5, it suffices to prove that $\phi := (\wp_\omega(u), v - \zeta_\omega(u))$ is algebraic over $\mathbb{C}(\psi(u,v))$. To prove this, just note that both $\mathbb{C}(\phi(u,v))$ and $\mathbb{C}(\psi(u,v))$ are subfields of $\mathbb{C}(\wp_\omega(u), v - \zeta_\omega(u), \wp'_\omega(u))$ and that the transcendence degree over $\mathbb{C}$ of the three fields is 2 because $\wp'_\omega(u)$ is algebraic over $\mathbb{C}(\wp_\omega(u))$.

Next, we consider the case of groups of the eighth type, so we have a locally $\mathbb{C}$-Nash group of the form

$$(\mathbb{C}^2, +, (\wp_\omega(u), e^v \widetilde{\sigma}_{\omega,\xi}(u))),$$

for some $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $\xi \notin \langle 1, \omega \rangle_{\mathbb{Q}}$. Again, we define the elliptic curve $E := \mathbb{C}/\Omega_E$ where $\Omega_E := \langle 1, \omega \rangle_{\mathbb{Z}}$.

Consider the functions

$$\Phi(u,v) := \frac{\sigma_\omega(u-\xi)}{\sigma_\omega(u)\sigma_\omega(\xi)} e^{v + u\zeta_\omega(\xi)} = \tfrac{1}{\sigma_\omega(\xi)} \widetilde{\sigma}_{\omega,\xi}(u) e^{v+u\zeta_\omega(\xi)},$$
$$F(u,v) := \frac{\wp'_\omega(u) + \wp'_\omega(\xi)}{\wp_\omega(u) - \wp_\omega(\xi)}.$$

Then, by [**17**, pag 32-34] (see also D. Caveny and R. Tubbs [**10**, §2]), we have that the map $\varphi : \mathbb{C}^2 \to \mathbb{P}^8$ given by

$$\begin{aligned}\varphi(u,v) = (1 : \wp_\omega(u) : \wp'_\omega(u) : \Phi(u,v) : \Phi(-u,-v) : \wp_\omega(u)\Phi(u,v) \\ : \wp_\omega(u)\Phi(-u,-v) : \Phi(u,v)F(u) : \Phi(-u,-v)F(-u))\end{aligned}$$

if $u \notin \Omega_E$ and

$$\varphi(u,v) = (0 : 0 : 1 : 0 : 0 : \tfrac{\sigma_\omega(u-\xi)}{\sigma_\omega(\xi)} e^{v+u\zeta_\omega(\xi)} : \tfrac{\sigma_\omega(-u-\xi)}{\sigma_\omega(\xi)} e^{-v-u\zeta_\omega(\xi)} : 0 : 0)$$

otherwise, is a parametrization of a quasi-projective subvariety G of $\mathbb{P}^8$. The addition in $\mathbb{C}^2$ induces an algebraic group structure on G and the kernel of φ is $\Omega := \{(\lambda, \xi\eta(\lambda) - \lambda\zeta_\omega(\xi) + 2\pi i m) \mid \lambda \in \Omega_E, m \in \mathbb{Z}\}$, which is a discrete subgroup of $\mathbb{C}^2$ of rank 3. Moreover, G is an extension of the elliptic curve E by $(\mathbb{C}^*, \cdot)$.

By Lemma 3.9, the map $\alpha(u,v) = (u, v + u\zeta_\omega(\xi))$ is an isomorphism from $(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$ to $(\mathbb{C}^2, +, (\wp_\omega(u), \Phi(u,v))$. Therefore, it is enough to prove that φ is a locally $\mathbb{C}$-Nash map when we equip $\mathbb{C}^2$ with the locally $\mathbb{C}$-Nash structure $(\mathbb{C}^2, +, (\wp_\omega(u), \Phi(u,v))$.

As before, recall that $(\mathbb{C}^2, +, (\wp_\omega(u), \Phi(u,v))$ is a notation that means that there exists $a \in \mathbb{C}$ such that $\phi_a(u,v) := (\wp_\omega(u+a), \Phi(u+a,v))$ is a chart of the identity of the locally $\mathbb{C}$-Nash group structure on $(\mathbb{C}^2, +)$ (see Remark 3.7). On the other hand, there is an open neighborhood W of the identity of G such that the projection $\pi : W \to \mathbb{C}^2 : (x_0 : \ldots : x_8) \mapsto (x_1/x_2, x_3/x_2)$ is a chart of the locally $\mathbb{C}$-Nash structure of G. By Proposition 2.9, it is enough to prove that $(\wp_\omega(u+a), \Phi(u+a,v))$ is algebraic over $\mathbb{C}(\psi(u,v))$, where

$\psi(u,v) := (\wp_\omega(u)(\wp'_\omega(u))^{-1}, \Phi(u,v)(\wp'_\omega(u))^{-1})$. In fact, by Corollary 3.5, it is enough to prove that $\phi := (\wp_\omega(u), \Phi(u,v))$ is algebraic over $\mathbb{C}(\psi(u,v))$. To prove this, just note that both $\mathbb{C}(\phi(u,v))$ and $\mathbb{C}(\psi(u,v))$ are subfields of $\mathbb{C}(\wp_\omega(u), \Phi(u,v), \wp'_\omega(u))$ and that the transcendence degree over $\mathbb{C}$ of the three fields is 2 because $\wp'_\omega(u)$ is algebraic over $\mathbb{C}(\wp_\omega(u))$. □

Obviously, we could infer Corollary 4.29 from Theorem 3.12 and the classification of two-dimensional complex abelian algebraic groups (see J.-P. Serre [**40**] and also P. Corvaja, D. Masser and U. Zannier [**12**]). However, we would like to stress the other direction: using Corollaries 2.16 and 4.29, we can provide a new proof with analytic methods of the classification of two-dimensional complex abelian algebraic groups.

Corollary 4.30. *Every two-dimensional abelian irreducible algebraic group is isogenous to one of the algebraic groups listed in Corollary 4.29.*

Proof. Let G be a two-dimensional abelian irreducible algebraic group. Then, by Corollary 2.15 and Proposition 2.13, its universal covering $\widetilde{G}$ is a connected two-dimensional abelian locally ℂ-Nash group. Then, by Corollary 4.29, we have that $\widetilde{G}$ is the universal covering of one of the algebraic groups listed in Corollary 4.29. In particular, we have a local isomorphism which is a locally ℂ-Nash map between G and one these groups, so the result follows from Corollary 2.16. □

As we showed in Remark 3.11, the classification in the abelian general, not necessarily simply connected, case relies on the automorphisms of the simply connected locally ℂ-Nash groups. To describe such groups, we need to introduce more notation. First, given a pair of lattices Ω_1 and Ω_2 of $(\mathbb{C}^2,+)$ with a common sublattice, we will associate two numbers $[\Omega_2 : \Omega_1]$ and $\mathfrak{qc}(\Omega_2,\Omega_1)$, see Definition 4.26. We recall that, given two complex numbers a and b, we denote by $\operatorname{diag}(a,b)$ the diagonal 2×2 matrix whose entries in the diagonal are a and b and that, given two subsets A and B, we denote $\operatorname{Diag}(A,B) := \{\operatorname{diag}(a,b) \mid a\in A, b\in B\}$. We also recall that, given $\omega\in\mathbb{C}\setminus\mathbb{R}$, we defined $K_\omega = \mathbb{Q}(\omega)$ if ω is quadratic over $\mathbb{Q}$ and $K_\omega := \mathbb{Q}$ otherwise. The isomorphisms and automorphisms of the case (9), which are universal coverings of abelian varieties which are not isomorphic to a direct product of elliptic curves, are beyond the objectives of this memoir.

Proposition 4.31. *Let $\omega,\omega_1,\omega_2\in\mathbb{C}\setminus\mathbb{R}$ and $\xi\in\mathbb{C}\setminus\langle 1,\omega\rangle_{\mathbb{Q}}$ and denote $\Omega = \langle 1,\omega\rangle_{\mathbb{Z}}$. Then,*

(1) $\operatorname{Aut}(\mathbb{C}^2,+,\mathrm{id}\times\mathrm{id}) = \mathrm{GL}_2(\mathbb{C})$.
(2) $\operatorname{Aut}(\mathbb{C}^2,+,\exp\times\mathrm{id}) = \operatorname{Diag}(\mathbb{Q}^*,\mathbb{C}^*)$.
(3) $\operatorname{Aut}(\mathbb{C}^2,+,\wp_\omega\times\mathrm{id}) = \operatorname{Diag}(K^*_\omega,\mathbb{C}^*)$.
(4) $\operatorname{Aut}(\mathbb{C}^2,+,\exp\times\exp) = \mathrm{GL}_2(\mathbb{Q})$.
(5) $\operatorname{Aut}(\mathbb{C}^2,+,\wp_\omega\times\exp) = \operatorname{Diag}(K^*_\omega,\mathbb{Q}^*)$.
(6.1) $\operatorname{Aut}(\mathbb{C}^2,+,\wp_{\omega_1}\times\wp_{\omega_2}) = \operatorname{diag}(1,\tau^{-1})\,\mathrm{GL}_2(K_{\omega_1})\operatorname{diag}(1,\tau)$
if there exists $\tau\in\mathbb{C}^*$ such that $\wp_{\omega_1}(\tau u)$ is algebraic over $\mathbb{C}(\wp_{\omega_2}(u))$, and
(6.2) $\operatorname{Aut}(\mathbb{C}^2,+,\wp_{\omega_1}\times\wp_{\omega_2}) = \operatorname{Diag}(K^*_{\omega_1},K^*_{\omega_2})$ *otherwise.*
(7) $\operatorname{Aut}(\mathbb{C}^2,+,(\wp_\omega(u),v-\zeta_\omega(u))) = \left\{ q\begin{pmatrix} 1 & 0\\ \mathfrak{qc}(\Omega,q\Omega) & [\Omega:q\Omega]q^{-2}\end{pmatrix} \;\middle|\; q\in K^*_\omega\right\}$.

(8) $\operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u))) = \left\{ q \begin{pmatrix} 1 & 0 \\ \xi\mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^* \right\}.$

Proof. If f is the chart of the identity that defines the locally $\mathbb{C}$-Nash structure then, by Lemma 3.9, we have that α is a locally $\mathbb{C}$-Nash automorphism if and only if $\alpha \in \mathrm{GL}_2(\mathbb{C})$ and $f \circ \alpha$ is algebraic over $\mathbb{C}(f)$. Let $\alpha(u, v) = (au + bv, cu + dv)$. As in Proposition 4.8, it is enough to check in each case for which $a, b, c, d \in \mathbb{C}$ the map α has the property that $f \circ \alpha$ is algebraic over $\mathbb{C}(f)$.

(1) If $f = \mathrm{id} \times \mathrm{id}$, clearly the property follows for all $\alpha \in \mathrm{GL}_2(\mathbb{C})$.

(2) If $f = \exp \times \mathrm{id}$ then $(e^{au+bv}, cu + dv)$ is algebraic over $\mathbb{C}(e^u, v)$. If we fix $u = 0$ then e^{bu} is algebraic over $\mathbb{C}(v)$, so that $b = 0$. Similarly, if we fix $v = 0$ then cu is algebraic over $\mathbb{C}(e^u)$, so that $c = 0$. Since $b = 0$, the restriction of α to the first coordinate of $\mathbb{C}^2$ is the map $u \mapsto au$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \exp)$. Also, since $c = 0$, the restriction of α to the second coordinate of $\mathbb{C}^2$ is the map $u \mapsto du$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \mathrm{id})$. Hence, by Proposition 4.8, we get that $a \in \mathbb{Q}^*$ and $d \in \mathbb{C}^*$. The converse is clear, so these are all the locally $\mathbb{C}$-Nash automorphisms.

(3) If $f = \wp_\omega \times \mathrm{id}$ then $(\wp_\omega(au+bv), cu+dv)$ is algebraic over $\mathbb{C}(\wp_\omega(u), v)$. If we fix $u = 0$ then $\wp_\omega(bv)$ is algebraic over $\mathbb{C}(v)$, so that $b = 0$. Similarly, if we fix $v = 0$ then cu is algebraic over $\mathbb{C}(\wp_\omega(u))$, so that $c = 0$. Since $b = 0$, the restriction of α to the first coordinate of $\mathbb{C}^2$ is the map $u \mapsto au$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \wp_\omega)$. Also, since $c = 0$, the restriction of α to the second coordinate of $\mathbb{C}^2$ is the map $u \mapsto du$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \mathrm{id})$. By Proposition 4.8, $a \in K_\omega^*$ and $d \in \mathbb{C}^*$. The converse is clear, so these are all the locally $\mathbb{C}$-Nash automorphisms.

(4) If $f = \exp \times \exp$ then (e^{au+bv}, e^{cu+dv}) is algebraic over $\mathbb{C}(e^u, e^v)$. If we fix $v = 0$ then e^{au} is algebraic over $\mathbb{C}(e^u)$. Hence, if $a \neq 0$ then, by Lemma 3.9, the map $u \mapsto au$ defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \exp)$ and, by Proposition 4.8, $a \in \mathbb{Q}^*$, so $a \in \mathbb{Q}$. Similarly, $b, c, d \in \mathbb{Q}$, and $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. The converse is clear, so these are all the locally $\mathbb{C}$-Nash automorphisms.

(5) If $f = \wp_\omega \times \exp$, $(\wp_\omega(au + bv), e^{cu+dv})$ is algebraic over $\mathbb{C}(\wp_\omega(u), e^v)$. If we fix $u = 0$ then $\wp_\omega(bv)$ is algebraic over $\mathbb{C}(e^v)$, so that $b = 0$. Similarly, if we fix $v = 0$ then cu is algebraic over $\mathbb{C}(\wp_\omega(u))$, so $c = 0$. Since $b = 0$, the restriction of α to the first coordinate of $\mathbb{C}^2$ is the map $u \mapsto au$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \wp_\omega)$. Also, since $c = 0$, the restriction of α to the second coordinate of $\mathbb{C}^2$ is the map $u \mapsto du$ and defines a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}, +, \exp)$. By Proposition 4.8, $a \in K_\omega^*$ and $d \in \mathbb{Q}^*$. The converse is clear, so these are all the locally $\mathbb{C}$-Nash automorphisms.

(6) We now compute $\operatorname{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ studying several cases. Let $\alpha(u, v) = (au + bv, cu + dv)$. We first compute the case where $\omega_1 = \omega_2$. If $\alpha \in \operatorname{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_1})$ then $(\wp_{\omega_1}(au+bv), \wp_{\omega_1}(cu+dv))$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), \wp_{\omega_1}(v))$. If we evaluate at $v = v_0$, for some $v_0 \notin \Omega_1 := \langle 1, \omega_1 \rangle_{\mathbb{Z}}$ then $\wp_{\omega_1}(au+bv_0)$ and $\wp_{\omega_1}(cu+dv_0)$ are algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$. By Corollary 3.5.(1), both $\wp_{\omega_1}(au)$ and $\wp_{\omega_1}(cu)$ are algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$, so that

$a, c \in K_{\omega_1}$. Similarly, if we evaluate at $u = u_0$, for some $u_0 \notin \Omega_1$, we obtain $b, d \in K_{\omega_1}$. Thus, $\alpha \in \mathrm{GL}_2(K_{\omega_1})$. For the converse, note that if $a, b, c, d \in K_{\omega_1}$, we obtain, since $\wp_{\omega_1}(u)$ admits an AAT, that $\wp_{\omega_1}(au+bv)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(au), \wp_{\omega_1}(bv))$, which in turn is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), \wp_{\omega_1}(v))$. Similarly, $\wp_{\omega_1}(cu+dv)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), \wp_{\omega_1}(v))$, as required.

(6.1) By hypothesis, there exists $\tau \in \mathbb{C}^*$ such that $\wp_{\omega_1}(\tau u)$ is algebraic over $\wp_{\omega_2}(u)$. Then, $\beta(u, v) = (u, \tau v)$ is an isomorphism from $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ to $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_1})$. Thus, $\alpha \in \mathrm{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ if and only if $\beta \circ \alpha \circ \beta^{-1} \in \mathrm{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_1})$. That is,

$$\mathrm{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2}) = \beta^{-1} \,\mathrm{GL}_2(K_{\omega_1})\beta,$$

as required.

We now check that $\mathrm{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ does not depend on the τ chosen. Indeed, if β' is also an isomorphim from $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ to $(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_1})$ then $\gamma := \beta' \circ \beta^{-1} \in \mathrm{GL}_2(K_{\omega_1})$ and, therefore,

$$\beta'^{-1} \,\mathrm{GL}_2(K_{\omega_1})\beta' = (\beta^{-1} \circ \gamma^{-1}) \,\mathrm{GL}_2(K_{\omega_1})(\gamma \circ \beta) = \beta^{-1} \,\mathrm{GL}_2(K_{\omega_1})\beta.$$

(6.2) By hypothesis, there is not $\tau \in \mathbb{C}^*$ such that $\wp_{\omega_1}(\tau u)$ is algebraic over $\wp_{\omega_2}(u)$. If $\alpha \in \mathrm{Aut}(\mathbb{C}^2, +, \wp_{\omega_1} \times \wp_{\omega_2})$ then $(\wp_{\omega_1}(au+bv), \wp_{\omega_2}(cu+dv))$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u), \wp_{\omega_2}(v))$. If we evaluate at $v = v_0$, for some $v_0 \notin \langle 1, \omega_2 \rangle_{\mathbb{Z}}$, then we obtain that $\wp_{\omega_1}(au)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$ and $\wp_{\omega_2}(cu)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$. In particular, $a \in K_{\omega_1}^*$ and $c = 0$ (otherwise we would have that $\wp_{\omega_1}(c^{-1}u)$ is algebraic over $\mathbb{C}(\wp_{\omega_2}(u))$, a contradiction). If we evaluate at $u = u_0$, for some $u_0 \notin \langle 1, \omega_1 \rangle_{\mathbb{Z}}$, then we obtain $b = 0$ and $d \in K_{\omega_2}^*$,as required. The converse is obvious.

(7) We first show that the maps showed in the statement are elements of $\mathrm{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), v - \zeta_\omega(u)))$. Indeed, suppose that ω is quadratic over $\mathbb{Q}$, so there exist $A, B \in \mathbb{Z}$ and $C \in \mathbb{N}^*$ such that $C\omega^2 + B\omega + A = 0$. Let $q \in K_\omega^*$. We can write $q = \frac{n}{cw+d}$, for certain $n \in \mathbb{N}^*$ and $c, d \in \mathbb{Z}$. If we define $a := Cd - cB$ and $b := -Ac$ then

$$\omega = \frac{a\omega + b}{C(c\omega + d)}.$$

Moreover, $ad - bc \neq 0$, because otherwise the equality above would provide that $\omega \in \mathbb{R}$, a contradiction. Finally, since $q = \frac{Cn}{C(cw_1+d)}$, by Lemma 4.18 we obtain that

$$\alpha(u, v) = \left(qu, q^{-1}\left(\mathfrak{c}(\Omega' : Cn\Omega') - \frac{\mathfrak{c}(\Omega, Cn\Omega')[\Omega', Cn\Omega']}{[\Omega, Cn\Omega']}\right)u + \frac{[\Omega' : Cn\Omega']}{[\Omega : Cn\Omega']}q^{-1}v\right)$$

where $\Omega' := q^{-1}\Omega$. Suppose now that ω is not quadratic over $\mathbb{Q}$ and let $q \in K_\omega^*$, so $q = \frac{n}{d}$ for some $n \in \mathbb{N}$ and $d \in \mathbb{Z}^*$. Clearly, $\omega = \frac{d\omega}{d}$, so that, by Lemma 4.18, we obtain

$$\alpha(u, v) = \left(qu, q^{-1}\left(\mathfrak{c}(\Omega' : n\Omega') - \frac{\mathfrak{c}(\Omega, n\Omega')[\Omega', n\Omega']}{[\Omega, n\Omega']}\right)u + \frac{[\Omega' : n\Omega']}{[\Omega : n\Omega']}q^{-1}v\right)$$

where $\Omega' := q^{-1}\Omega$. By Proposition 4.26, in both cases we can write $\alpha(u, v) = (qu, q^{-1}\mathfrak{qc}(q^{-1}\Omega, \Omega)u + [q^{-1}\Omega : \Omega]q^{-1}v)$. Clearly, $[q^{-1}\Omega : \Omega] = [\Omega : q\Omega]$ and, by Lemma 4.27.(2), $\mathfrak{qc}(q^{-1}\Omega, \Omega) = q^2\mathfrak{qc}(\Omega, q\Omega)$, so we get

$$\alpha(u, v) = q\big(u, \mathfrak{qc}(\Omega, q\Omega)u + [\Omega : q\Omega]q^{-2}v\big).$$

Next, let $\alpha(u,v) = (au+bv, cu+dv) \in \operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), v - \zeta_\omega(u)))$. By the claim in the proof of Lemma 4.18 (with $\xi = 1$), we have that $b = 0$ and $\wp_\omega(au)$ is algebraic over $\mathbb{C}(\wp_\omega(u))$. By Proposition 4.8, $a \in K_\omega^*$. So it is enough to prove that, for each $a \in K_\omega^*$, there exists at most one automorphism of the form $\alpha(u,v) = (au, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$. That is, an automorphism of the form $(u, cu+dv)$ must be the identity, which follows from the claim of the proof of Proposition 4.26.

(8) First, we show $q\begin{pmatrix} 1 & 0 \\ \xi\mathfrak{q}\mathfrak{c}(\Omega, q\Omega) & 1\end{pmatrix} \in \operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$ for each $q \in \mathbb{Q}^*$. Take $q_1, q_2 \in \mathbb{Z}^*$ coprimes such that $q = q_1 q_2^{-1}$. By Corollary 4.25, we have that both

$$\alpha_1(u,v) := q_1^{-1}(u, -\xi\mathfrak{c}(\Omega, q_1\Omega)u + v)$$
$$\alpha_2(u,v) := q_2^{-1}(u, -\xi\mathfrak{c}(\Omega, q_2\Omega)u + v)$$

belong to $\operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$. Therefore, by Lemma 4.27,

$$(\alpha_1^{-1} \circ \alpha_2)(u,v) = q(u, \xi\mathfrak{q}\mathfrak{c}(\Omega, q\Omega)u + v)$$

also belongs to $\operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$, as required.

Next, let $\alpha(u,v) = (au+bv, cu+dv) \in \operatorname{Aut}(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$. By Lemma 4.22, $a \in K_\omega^*$, $b = 0$ and $d \in \mathbb{Q}^*$. If ω is not quadratic then $a \in \mathbb{Q}^*$ because $K_\omega^* = \mathbb{Q}^*$. Moreover:

Claim. Let ω be quadratic over $\mathbb{Q}$. If $\alpha(u,v) := (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$ is an automorphism of $(\mathbb{C}^2, +, (\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u)))$ then $a \in \mathbb{Q}^*$.

Proof of the claim. Recall that, by hypothesis, $\xi \notin \langle 1, \omega\rangle_{\mathbb{Q}} = K_\omega$. Suppose by contradiction that $a \in K_\omega \setminus \mathbb{Q}^*$. By Lemma 4.22, both $b = 0$ and $\wp_\Omega(au)$ is algebraic over $\mathbb{C}(\wp_\Omega(u))$. In particular, $a \neq 0$ and $\wp_{a^{-1}\Omega}(u)$ is algebraic over $\mathbb{C}(\wp_\Omega(u))$. By Lemma 4.6, there exists a sublattice Ω' of both $a^{-1}\Omega$ and Ω. By Lemma 4.20, there exists $\alpha_1(u,v) := (u, c_1u + d_1v)$ in $\mathrm{GL}_2(\mathbb{C})$ such that

$$(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)) \circ \alpha_1 \text{ is algebraic over } \mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u)).$$

Similarly, there exists $\alpha_2(u,v) := (u, c_2u + d_2v)$ in $\mathrm{GL}_2(\mathbb{C})$ such that

$$(\wp_{a^{-1}\Omega}(u), e^v\widetilde{\sigma}_{a^{-1}\Omega,a^{-1}\xi}(u)) \circ \alpha_2 \text{ is algebraic over } \mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',a^{-1}\xi}(u)).$$

Let $\beta(u,v) := (u, cu+dv)$. Since $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)) \circ \alpha$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$,

$$(\wp_{a^{-1}\Omega}(u), e^v\widetilde{\sigma}_{a^{-1}\Omega,a^{-1}\xi}(u)) \circ \beta \text{ is algebraic over } \mathbb{C}(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u)).$$

Therefore, we deduce that

$$\mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',a^{-1}\xi}(u)) \circ \alpha_2^{-1} \circ \beta \circ \alpha_1 \text{ is algebraic over } \mathbb{C}(\wp_{\Omega'}(u), e^v\widetilde{\sigma}_{\Omega',\xi}(u)).$$

Note that the isomorphism $\alpha_2^{-1} \circ \beta \circ \alpha_1$ is of the form $(u, c_4u + d_4v) \in \mathrm{GL}_2(\mathbb{C})$. Thus, by Lemma 4.22, there exists $d' \in \mathbb{Q}^*$ such that

$$a^{-1}\xi - d'\xi = (a^{-1} - d')\xi \in \Omega' < \Omega < K_\omega = \langle 1, \omega\rangle_{\mathbb{Q}} .$$

Recall that $a \in K_\omega^*$, so that $a^{-1} - d' \in K_\omega$. Moreover, by hypothesis, $a \notin \mathbb{Q}^*$ and, therefore, $a^{-1} - d' \neq 0$. We deduce that $\xi \in K_\omega$, which is the desired contradiction. $\square$

Finally, as we did in the case (7), it is enough to show that the identity map is the only automorphism of the form $(u, cu+dv)$, for $c \in \mathbb{C}$ and $d \in \mathbb{Q}^*$. Indeed, if $(\wp_\omega(u), e^{cu+dv}\widetilde{\sigma}_{\omega,\xi}(u))$ is algebraic over $\mathbb{C}(\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u))$ then, by Lemma 4.22, $\xi - d\xi \in \Omega$. Since $d \in \mathbb{Q}^*$ and $\xi \notin \langle 1, \omega\rangle_{\mathbb{Q}}$, we have $d = 1$. As e^{cu} is algebraic over $\mathbb{C}(\wp_\omega(u), e^v\widetilde{\sigma}_{\omega,\xi}(u))$ and, hence, over $\mathbb{C}(\wp_\omega(u))$, which implies $c = 0$. $\square$

The abelian general, not necessarily simply connected, case follows substituting (I) and (II) in Proposition 3.10 (and (a) and (b) of Remark 3.11) with the corresponding calculations of the two-dimensional case, namely Theorem 4.28 and Proposition 4.31.

Chapter 5

Abelian locally Nash groups of dimensions 1 and 2

In this chapter, we prove the second main result of this thesis: the classification of two-dimensional abelian locally Nash groups. By Proposition 3.10 and Remark 3.11, the classification reduces to those simply connected ones, and the description of their automorphisms groups. By Theorem 3.8, every simply connected abelian locally Nash group is of the form $(\mathbb{R}^n, +, f)$, where $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is an invariant meromorphic map admitting an AAT. By Remark 3.14, $(\mathbb{C}^n, +, f)$ is a locally $\mathbb{C}$-Nash group and, for $n = 1$ and $n = 2$, we can use the classifications obtained Chapter 4. That is, there exists $\alpha \in \mathrm{GL}_n(\mathbb{C})$ which is an isomorphism from $(\mathbb{C}^n, +, f)$ to a certain $(\mathbb{C}^n, +, g)$, where g is one of the maps listed in Theorem 4.7 (one-dimensional case) or Theorem 2 (two-dimensional case). Note that such maps g are not invariant in general and, therefore, we cannot compare $(\mathbb{C}^n, +, g)$ with a locally Nash group. However, the fact that f is invariant will impose some conditions on $g \circ \alpha$ that will lead us to other invariant meromorphic maps admitting an AAT. On the other hand, such a map g can lead to different invariant maps. For example, the groups $(\mathbb{R}, +, \exp)$ and $(\mathbb{R}, +, \sin)$ are not isomorphic as locally Nash groups although $(\mathbb{C}, +, \exp)$ and $(\mathbb{C}, +, \sin)$ are isomorphic as locally $\mathbb{C}$-Nash groups.

The chapter is divided as follows. In Section 1, we give an alternative proof of the classification of one-dimensional simply connected locally Nash groups (Fact 5.6) and we study their automorphism groups (Proposition 5.7). In Section 2, we give a classification of two-dimensional simply connected abelian locally $\mathbb{C}$-Nash groups (Theorem 5.10) and describe their automorphism groups (Proposition 5.11).

1. One-dimensional locally Nash groups

A classification of one-dimensional simply connected locally Nash groups was given by Madden and Stanton in [**26**] (see also [**25**]). In this section, we provide an alternative proof of such classification using the techniques we have developed in this memoir.

We first study for which lattices $\Omega < (\mathbb{C},+)$ the meromorphic function $\wp_\Omega$ is an invariant meromorphic function (we also do the same for the Weierstrass ζ_Ω and σ_Ω functions, this will be used later, in the two-dimensional case).

LEMMA 5.1. *Let Ω be a lattice of $(\mathbb{C},+)$ and $\xi \in \mathbb{C}^*$. Then, $\wp_{\overline{\Omega}}(u) = \overline{\wp_\Omega(\overline{u})}$, $\zeta_{\overline{\Omega}}(u) = \overline{\zeta_\Omega(\overline{u})}$, $\sigma_{\overline{\Omega}}(u) = \overline{\sigma_\Omega(\overline{u})}$ and $\widetilde{\sigma}_{\overline{\Omega},\overline{\xi}}(u) = \overline{\widetilde{\sigma}_{\Omega,\xi}(\overline{u})}$. Consequently, $\wp_\Omega$, ζ_Ω, σ_Ω are invariant meromorphic functions if and only if $\Omega = \overline{\Omega}$.*

PROOF. We note that

$$\overline{\wp_\Omega(u)} = \overline{\frac{1}{u^2} + \sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right)} = \frac{1}{\overline{u}^2} + \sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{(\overline{u}-\overline{\omega})^2} - \frac{1}{\overline{\omega}^2}\right).$$

Therefore,

$$\overline{\wp_\Omega(u)} = \frac{1}{\overline{u}^2} + \sum_{\omega\in\overline{\Omega}\setminus\{0\}} \left\{\frac{1}{(\overline{u}-\omega)^2} - \frac{1}{\omega^2}\right\}.$$

For ζ_Ω, we note that, by [**11**, Ch.IV, equation (1.1)],

$$\overline{\zeta_\Omega(u)} = \overline{\frac{1}{u} + \sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right)} = \frac{1}{\overline{u}} + \sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{\overline{u}-\overline{\omega}} + \frac{1}{\overline{\omega}} + \frac{\overline{u}}{\overline{\omega}^2}\right).$$

Therefore,

$$\overline{\zeta_\Omega(u)} = \frac{1}{\overline{u}} + \sum_{\omega\in\overline{\Omega}\setminus\{0\}} \left(\frac{1}{\overline{u}-\omega} + \frac{1}{\omega} + \frac{\overline{u}}{\omega^2}\right).$$

Similarly, by [**11**, Ch.IV, equation (2.5)],

$$\overline{\sigma_\Omega(u)} = \overline{u \prod_{\omega\in\Omega\setminus\{0\}} \left(1-\frac{u}{\omega}\right)\exp\left(\frac{u}{\omega}+\frac{u^2}{2\omega^2}\right)} = \overline{u} \prod_{\omega\in\Omega\setminus\{0\}} \left(1-\frac{\overline{u}}{\overline{\omega}}\right)\exp\left(\frac{\overline{u}}{\overline{\omega}}+\frac{\overline{u}^2}{2\overline{\omega}^2}\right).$$

Therefore,

$$\overline{\sigma_\Omega(u)} = \overline{u} \prod_{\omega\in\overline{\Omega}\setminus\{0\}} \left(1-\frac{\overline{u}}{\omega}\right)\exp\left(\frac{\overline{u}}{\omega}+\frac{\overline{u}^2}{2\omega^2}\right).$$

Finally,

$$\overline{\widetilde{\sigma}_{\Omega,\xi}(u)} = \frac{\overline{\sigma_\Omega(u-\xi)}}{\overline{\sigma_\Omega(u)}} = \frac{\sigma_{\overline{\Omega}}(\overline{u-\xi})}{\sigma_{\overline{\Omega}}(\overline{u})} = \frac{\sigma_{\overline{\Omega}}(\overline{u}-\overline{\xi})}{\sigma_{\overline{\Omega}}(\overline{u})} = \widetilde{\sigma}_{\overline{\Omega},\overline{\xi}}(\overline{u}).$$

For the second part of the statement, recall that a meromorphic function f is invariant if and only if $\overline{f(\overline{u})} = f(u)$. □

LEMMA 5.2. *Let Λ be a lattice of $(\mathbb{C},+)$. Let $g : \mathbb{C} \dashrightarrow \mathbb{C}$ be an invariant meromorphic function such that Λ_g is a discrete subgroup of $(\mathbb{C},+)$ and g is algebraic over $\mathbb{C}(\wp_\Lambda)$. Then, there exists an invariant lattice $\Lambda' < \Lambda$ such that g is algebraic over $\mathbb{C}(\wp_{\Lambda'})$.*

PROOF. Since Λ is a lattice, Λ_g is an invariant lattice by Lemmas 3.13.(3) and 3.15. Hence, by Corollary 3.16, there exists an invariant lattice Λ' of $(\mathbb{C},+)$ such that $\Lambda' < \Lambda$ and $\Lambda' < \Lambda_g$. On the other hand, g is algebraic over $\mathbb{C}(\wp_\Lambda)$ and $\wp_\Lambda$ is algebraic over $\mathbb{C}(\wp_{\Lambda'})$, by Lemma 4.5, so g is algebraic over $\mathbb{C}(\wp_{\Lambda'})$. □

LEMMA 5.3. *Let Λ be a lattice of $(\mathbb{C},+)$. If $\wp_\Lambda$ and $\wp_{\overline{\Lambda}}$ are algebraically dependent then $\Lambda \cap \overline{\Lambda}$ is an invariant lattice.*

PROOF. Clearly, $\Lambda \cap \overline{\Lambda}$ is invariant, so it is enough to show that it is a lattice. Indeed, it is a (discrete) subgroup of both Λ and $\overline{\Lambda}$ and, by Corollary 3.16, there exists a lattice $\Lambda_1 < \Lambda \cap \overline{\Lambda}$, as required. □

Some of the locally Nash group structures on $(\mathbb{R},+)$ will be given by Weierstrass $\wp$-functions over lattices Λ. We will use the notation $(\mathbb{R},+,\wp_\Lambda)$ of the introduction (see Remark 3.7).

REMARK 5.4. Note that a non-trivial invariant discrete subgroup Λ of $(\mathbb{C},+)$ is of rank 1 if it is either of the form $\langle a\rangle_{\mathbb{Z}}$ or $\langle ia\rangle_{\mathbb{Z}}$, for some $a \in \mathbb{R}^*$; and it is of rank 2 if it is has a finite index subgroup of the form $\langle a, bi\rangle_{\mathbb{Z}}$, for some $a, b \in \mathbb{R}^*$. Indeed, since Λ is invariant we must have $\overline{\lambda} \in \Lambda$, for any $\lambda \in \Lambda$. The only special case is when $\Lambda = \langle \lambda, \overline{\lambda}\rangle_{\mathbb{Z}}$, with $\lambda = a + ib$ with both $a, b \neq 0$. Then, $\langle 2a, 2ib\rangle_{\mathbb{Z}}$ is the finite index subgroup of Λ.

FACT 5.5 ([**26**, Theorem 2]). *Let $a, b \in \mathbb{R}^*$ and let $\Lambda_1 := \langle 1, ia\rangle_{\mathbb{Z}}$ and $\Lambda_2 := \langle 1, ib\rangle_{\mathbb{Z}}$. Then, $(\mathbb{R},+,\wp_{\Lambda_1})$ and $(\mathbb{R},+,\wp_{\Lambda_2})$ are isomorphic if and only if $ab^{-1} \in \mathbb{Q}$.*

PROOF. Firstly, suppose that there are $m, n \in \mathbb{Z}^*$ such that $mn^{-1} = ab^{-1}$. Let $\Lambda := \langle 1, ina\rangle_{\mathbb{Z}}$. Hence, both $\Lambda < \Lambda_1$ and $\Lambda < \Lambda_2$. So, by Lemma 4.5, $\wp_\Lambda$ is algebraic over both $\mathbb{C}(\wp_{\Lambda_1})$ and $\mathbb{C}(\wp_{\Lambda_2})$. Hence, $\wp_{\Lambda_1}$ is algebraic over $\mathbb{C}(\wp_{\Lambda_2})$. In particular, $\wp_{\Lambda_1}$ is algebraic over $\mathbb{R}(\wp_{\Lambda_2})$. So, by Lemma 3.9, $(\mathbb{R},+,\wp_{\Lambda_1})$ and $(\mathbb{R},+,\wp_{\Lambda_2})$ are isomorphic.

Suppose now that $(\mathbb{R},+,\wp_{\Lambda_1})$ and $(\mathbb{R},+,\wp_{\Lambda_2})$ are isomorphic. By Lemma 3.9, there exists $\alpha \in \mathrm{GL}_1(\mathbb{R})$ such that

$$\wp_{\Lambda_2} \circ \alpha \text{ is algebraic over } \mathbb{R}(\wp_{\Lambda_1}).$$

Let c denote the unique element of $\mathbb{R}^*$ such that

$$\alpha : \mathbb{R} \to \mathbb{R} : x \mapsto cx.$$

Let $\Lambda_2' := \alpha^{-1}(\Lambda_2)$. Then, $\Lambda_2' = \langle c^{-1}, ibc^{-1}\rangle_{\mathbb{Z}}$. We note that Λ_2' is the group of periods of $\wp_{\Lambda_2} \circ \alpha$. By Corollary 3.16, there exists an invariant lattice Λ of $(\mathbb{C},+)$ such that both $\Lambda < \Lambda_1$ and $\Lambda < \Lambda_2'$. By Remark 5.4, we may assume that there exist $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that

$$\Lambda = \langle n_1, n_2 ia\rangle_{\mathbb{Z}} = \langle m_1 c^{-1}, m_2 ibc^{-1}\rangle_{\mathbb{Z}}.$$

Thus, $m_1 c^{-1} = pn_1$, for some $p \in \mathbb{Z}$, so $c \in \mathbb{Q}$. Also $n_2 ia = pm_2 ibc^{-1}$, for some $p \in \mathbb{Z}$, so $ab^{-1} \in \mathbb{Q}$. □

Now we prove [**26**, Theorem 1] from a different point of view that involves ranks of lattices. We will use the notation $(\mathbb{R},+,f)$, see Remark 3.7 (in particular we recall that the map associated to a chart of the identity can be a translate of f).

FACT 5.6 ([**26**, Theorem 1]). (I) *Every simply connected one-dimensional locally Nash group is isomorphic to a group of one and only one of the following types:*

(1) $(\mathbb{R},+,\mathrm{id})$.
(2) $(\mathbb{R},+,\exp)$.

(3) $(\mathbb{R}, +, \sin)$.
(4) $(\mathbb{R}, +, \wp_\Lambda)$, *where* $\Lambda = \langle 1, ia \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$.

(II) $(\mathbb{R}, +, \wp_{\langle 1, ia \rangle_{\mathbb{Z}}})$ *and* $(\mathbb{R}, +, \wp_{\langle 1, ib \rangle_{\mathbb{Z}}})$ *are isomorphic if and only if* $a/b \in \mathbb{Q}$.

Proof. *(I)* We first note that, by Lemma 3.6, each of the four cases are indeed locally Nash groups. Every connected analytic group of dimension 1 is abelian, so, by Theorem 3.8, every simply connected one-dimensional locally Nash group is isomorphic to some $(\mathbb{R}, +, f)$, where $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is an invariant meromorphic function admitting an AAT. We are in the hypothesis of Fact 4.2 and, therefore, there exists $\alpha \in \mathrm{GL}_1(\mathbb{C})$ such that f is algebraic over $\mathbb{C}(g \circ \alpha)$, where $g : \mathbb{C} \dashrightarrow \mathbb{C}$ is either id or exp or $\wp_\Lambda$, for some lattice Λ of $(\mathbb{C}, +)$. Let $c \in \mathbb{C}^*$ be such that $\alpha : \mathbb{C} \to \mathbb{C} : u \mapsto cu$. We note that, by Lemma 3.13.(5), Λ_f is an invariant discrete subgroup of $(\mathbb{C}, +)$. We also note that, by Lemma 3.13.(6),

$$\operatorname{rank} \Lambda_{\mathrm{id} \circ \alpha} = 0, \quad \operatorname{rank} \Lambda_{\exp \circ \alpha} = 1, \quad \text{or} \quad \operatorname{rank} \Lambda_{\wp_\Lambda \circ \alpha} = 2.$$

Case I: $\operatorname{rank} \Lambda_f = 0$. Then, by Lemma 3.15, $\operatorname{rank} \Lambda_{g \circ \alpha} = 0$. So f is algebraic over $\mathbb{C}(c \cdot \mathrm{id}) = \mathbb{C}(\mathrm{id})$ and, therefore, by Lemma 3.9, $(\mathbb{R}, +, f)$ and $(\mathbb{R}, +, \mathrm{id})$ are isomorphic, what gives us (1) in the statement of the theorem.

Case II: $\operatorname{rank} \Lambda_f = 1$. Then, by Lemma 3.15, $\operatorname{rank} \Lambda_{g \circ \alpha} = 1$. So, $g = \exp$ and hence f is algebraic over $\mathbb{C}(\exp \circ \alpha)$. Since Λ_f is an invariant discrete subgroup of $(\mathbb{C}, +)$, by Lemma 3.15 $\Lambda_{\exp \circ \alpha}$ is also an invariant discrete subgroup of $(\mathbb{C}, +)$. By Remark 5.4, there exists $a \in \mathbb{R}^*$ such that either $\Lambda_{g \circ \alpha} = \langle a \rangle_{\mathbb{Z}}$ or either $\Lambda_{g \circ \alpha} = \langle ia \rangle_{\mathbb{Z}}$.
Subcase II.1: $\Lambda_{g \circ \alpha} = \langle ia \rangle_{\mathbb{Z}}$. In this case, $f(u)$ is algebraic over $\mathbb{C}(e^{2\pi u/a})$. Since both $f(u)$ and $e^{2\pi u/a}$ are invariant meromorphic functions, we get by Lemma 3.9 that $(\mathbb{R}, +, f)$ and $(\mathbb{R}, +, e^{2\pi x/a})$ are isomorphic. Let $\widetilde{\alpha}(x) := ax/2\pi \in \mathrm{GL}_1(\mathbb{R})$. Then, again by Lemma 3.9 applied to $\widetilde{\alpha}$, we deduce that $(\mathbb{R}, +, \exp)$ and $(\mathbb{R}, +, e^{2\pi x/a})$ are isomorphic. So $(\mathbb{R}, +, f)$ is isomorphic to $(\mathbb{R}, +, \exp)$, what gives us (2) in the statement of the theorem.
Subcase II.2: $\Lambda_{g \circ \alpha} = \langle a \rangle_{\mathbb{Z}}$. In this case, $f(u)$ is algebraic over $\mathbb{C}(e^{2\pi i u/a})$. Hence, $f(u)$ is algebraic over $\mathbb{R}(\sin(2\pi u/a))$. Hence, applying Lemma 3.9, we deduce that $(\mathbb{R}, +, f)$ and $(\mathbb{R}, +, \sin(2\pi x/a))$ are isomorphic. Again, by Lemma 3.9 applied to $\widetilde{\alpha}$ above, we get that $(\mathbb{R}, +, \sin)$ and $(\mathbb{R}, +, \sin(2\pi x/a))$ are isomorphic. So $(\mathbb{R}, +, f)$ is isomorphic to $(\mathbb{R}, +, \sin)$, what gives us (3) in the statement of the theorem.

Case 3: $\operatorname{rank} \Lambda_f = 2$. Then, by Lemma 3.15, $\operatorname{rank} \Lambda_{g \circ \alpha} = 2$. So there exists a lattice Λ of $(\mathbb{C}, +)$ such that $g = \wp_\Lambda$ and, hence, $f(u)$ is algebraic over $\mathbb{C}(\wp_\Lambda(cu))$. By Lemma 5.2 and since $\wp_\Lambda(cu) = c^{-2} \wp_{c^{-1}\Lambda}(u)$, f is algebraic over $\mathbb{C}(\wp_\Lambda)$, for some invariant lattice Λ of $(\mathbb{C}, +)$. Moreover, by Lemma 4.5 and Remark 5.4, we may assume that Λ is of the form $\langle a, ib \rangle_{\mathbb{Z}}$, for some $a, b \in \mathbb{R}^*$. Hence, applying Lemma 3.9, we deduce that $(\mathbb{R}, +, f)$ and $(\mathbb{R}, +, \wp_\Lambda)$ are isomorphic. Let $\Lambda' := \langle 1, ib/a \rangle_{\mathbb{Z}}$ and let $\widetilde{\alpha}(x) := a^{-1}x \in \mathrm{GL}_1(\mathbb{R})$. We note that $\wp_{\Lambda'}(\widetilde{\alpha}(x)) = a^2 \wp_\Lambda(x)$ and, therefore, by Lemma 3.9 applied to $\widetilde{\alpha}$, we deduce that $(\mathbb{R}, +, \wp_\Lambda)$ and $(\mathbb{R}, +, \wp_{\Lambda'})$ are isomorphic. So, in this case, $(\mathbb{R}, +, f)$ is isomorphic to $(\mathbb{R}, +, \wp_{\Lambda'})$, where $\Lambda' = \langle 1, ia \rangle_{\mathbb{Z}}$, for some $a \in \mathbb{R}^*$. This gives us (4) in the statement of the theorem.

Now, we show that the four types of groups considered are not isomorphic. By Proposition 3.17, the only ones that can be isomorphic are of the type (2) and (3) or both of the type (4). Suppose $(\mathbb{R},+,\exp)$ and $(\mathbb{R},+,\sin)$ are isomorphic. Then, by Lemma 3.9, there exists $\alpha \in \mathrm{GL}_1(\mathbb{R})$ such that $e^{\alpha(x)}$ is algebraic over $\mathbb{C}(\sin(x))$. Since the periods of e^x are imaginary, the periods of $\sin(x)$ are real numbers and α cannot map imaginary numbers into real numbers, this contradicts Lemma 3.15.

(II) The isomorphism classes of groups of type (4) follows from Fact 5.5. $\square$

Now, we compute the automorphism groups of the locally Nash groups of Fact 5.6. We recall that, given $\omega \in \mathbb{C} \setminus \mathbb{R}$, we denote by $\wp_\omega$ the function $\wp_{\langle 1,\omega\rangle_{\mathbb{Z}}}$.

Proposition 5.7. *Let* $\omega \in i\mathbb{R}^*$. *Then,* $\mathrm{Aut}(\mathbb{R},+,\exp)$, $\mathrm{Aut}(\mathbb{R},+,\sin)$ *and* $\mathrm{Aut}(\mathbb{R},+,\wp_\omega)$ *are* $\mathrm{GL}_1(\mathbb{Q})$ *and* $\mathrm{Aut}(\mathbb{R},+,\mathrm{id})$ *is* $\mathrm{GL}_1(\mathbb{R})$.

Proof. If $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is an invariant meromorphic function then, by Lemma 3.9, we have that α is a locally Nash automorphism of $(\mathbb{R},+,f)$ if and only if $\alpha \in \mathrm{GL}_1(\mathbb{R})$ and α is a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C},+,f)$. So the only case that is not trivial (see Proposition 4.8) is the one of $(\mathbb{R},+,\sin(u))$. We note that, since $\sin(u)$ is algebraic over $\mathbb{C}(e^{iu})$, the identity map is an isomorphism from $(\mathbb{C},+,\sin(u))$ to $(\mathbb{C},+,e^{iu})$. By Proposition 4.8, it follows that the group of automorphisms of $(\mathbb{C},+,e^{iu})$ is $\mathrm{GL}_1(\mathbb{Q})$, as required. $\square$

2. Two-dimensional abelian locally Nash groups

Finally, we consider abelian two-dimensional locally Nash groups. As we have already mentioned in the introduction to this chapter, it remains to classify the locally Nash group structures on $(\mathbb{R}^2,+)$ and describe their automorphism groups.

Lemma 5.8. *Let* Ω_1, Ω_2 *be invariant sublattices of* $(\mathbb{C},+)$ *having a common sublattice. Then,* $\mathfrak{qc}(\Omega_1,\Omega_2) \in \mathbb{R}$.

Proof. We first assume that $\Omega_1 < \Omega_2$. Then, it is possible to choose $a_1,\dots,a_n \in \Omega_2$ such that, for each $i \in \{1,\dots,n\}$, there exists $j \in \{1,\dots,n\}$ such that $\Omega_1 + a_i = \Omega_1 + \overline{a_j}$. Indeed, take $b_1,\dots,b_n$ such that $\Omega_2 := \bigcup_{i=1}^n \Omega_1 + b_i$. Fix $i \in \{1,\dots,n\}$. If $\Omega_1 + \overline{b_i} = \Omega_1 + b_i$ then we can take $a_i := b_i$. Otherwise, there exists $j \in \{1,\dots,n\}$ such that $\Omega_1 + \overline{b_i} = \Omega_1 + b_j$. So we can take $a_i := b_i$ and $a_j := b_j$. The claim is proved repeating this process until we have defined all the $a_1,\dots,a_n$.

Thus, by Lemma 4.3,

$$0 = \mathfrak{c} + \sum_{i=1}^n \wp_{\Omega_1}(u+a_i) - \wp_{\Omega_2}(u).$$

By Lemmas 3.4 and 5.1 and by our choice of $a_1,\dots,a_n$,

$$0 = \bar{\mathfrak{c}} + \sum_{i=1}^n \wp_{\Omega_1}(u+\overline{a_i}) - \wp_{\Omega_2}(u) = \bar{\mathfrak{c}} + \sum_{i=1}^n \wp_{\Omega_1}(u+a_i) - \wp_{\Omega_2}(u)$$

and, hence, $\mathfrak{c}-\overline{\mathfrak{c}}=0$, so $\mathfrak{c}\in\mathbb{R}$. Therefore, $\mathfrak{qc}(\Omega_2,\Omega_1)=\mathfrak{c}(\Omega_2,\Omega_1)=\mathfrak{c}\in\mathbb{R}$, as required.

Finally, note that if Ω_1 and Ω_2 have a common sublattice Ω then, by Lemma 4.6.(1) and Corollary 3.16, we may assume that Ω is also an invariant sublattice. Since both $\mathfrak{c}(\Omega_2,\Omega),\mathfrak{c}(\Omega_1,\Omega)\in\mathbb{R}$ and by definition

$$\mathfrak{qc}(\Omega_2,\Omega_1):=\mathfrak{c}(\Omega_2,\Omega)-\frac{[\Omega_2:\Omega]}{[\Omega_1:\Omega]}\mathfrak{c}(\Omega_1,\Omega),$$

we get that $\mathfrak{qc}(\Omega_2,\Omega_1)\in\mathbb{R}$. □

The next lemma will simplify the proofs of our classification result.

Lemma 5.9. *Let $\alpha(u,v):=(au+bv,cu+dv)\in\mathrm{GL}_2(\mathbb{C})$ be an analytic automophism of $\mathbb{C}^2$, $\xi\in\mathbb{C}$ and Ω a lattice of $(\mathbb{C},+)$. We have,*

(1) *If $\overline{c}u+\overline{d}v$ is algebraic over $\mathbb{C}(cu+dv)$ then there exist $c',d'\in\mathbb{R}$ such that $cu+dv$ is algebraic over $\mathbb{C}(c'u+d'v)$ and $(au+bv,c'u+d'v)\in\mathrm{GL}_2(\mathbb{C})$.*
(2) *If $\overline{c}u+\overline{d}v-\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(cu+dv-\zeta_\Omega(au+bv))$ then there exist $c',d'\in\mathbb{R}$ such that $cu+dv-\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(c'u+d'v-\zeta_\Omega(au+bv))$ and $(au+bv,c'u+d'v)\in\mathrm{GL}_2(\mathbb{C})$.*
(3) *If $e^{\overline{a}u+\overline{b}v}$ is algebraic over $\mathbb{C}(e^{au+bv})$ then there exist $a',b'\in\mathbb{R}$ and $\delta\in\{1,i\}$ such that e^{au+bv} is algebraic over $\mathbb{C}(e^{\delta(a'u+b'v)})$ and $(a'u+b'v,cu+dv)\in\mathrm{GL}_2(\mathbb{C})$.*
(4) *If $(e^{\overline{a}u+\overline{b}v},e^{\overline{c}u+\overline{d}v})$ is algebraic over $\mathbb{C}(e^{au+bv},e^{cu+dv})$ then there exist $a',b',c',d'\in\mathbb{R}$ and $\delta_1,\delta_2\in\{1,i\}$ such that (e^{au+bv},e^{cu+dv}) is algebraic over $\mathbb{C}(e^{\delta_1(a'u+b'v)},e^{\delta_2(c'u+d'v)})$ and $(a'u+b'v,c'u+d'v)\in\mathrm{GL}_2(\mathbb{R})$.*
(5) *If $e^{\overline{c}u+\overline{d}v}\widetilde{\sigma}_{\Omega,\overline{\xi}}(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv),e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$ then there exist $c',d',\xi'\in\mathbb{R}$, and $\delta\in\{1,i\}$ such that both $(\wp_\Omega(au+bv),e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv),e^{\delta(c'u+d'v)}\widetilde{\sigma}_{\Omega,\delta\xi'}(au+bv))$ and $(au+bv,c'u+d'v)\in\mathrm{GL}_2(\mathbb{C})$.*

Proof. (1) Both $(c+\overline{c})u+(d+\overline{d})v$ and $(c-\overline{c})u+(d-\overline{d})v$ are algebraic over $\mathbb{C}(cu+dv)$. If $(c+\overline{c})u+(d+\overline{d})v$ is not zero, we take $c':=(c+\overline{c})$, $d':=(d+\overline{d})$. Otherwise, we take $c':=i(c-\overline{c})$, $d':=i(d-\overline{d})$.

Now, we show that $(au+bv,c'u+d'v)\in\mathrm{GL}_2(\mathbb{C})$. We note that $(au+bv,cu+dv)$ is algebraic over $\mathbb{C}(au+bv,c'u+d'v)$. Since the transcendence degree of $(au+bv,cu+dv)$ over $\mathbb{C}$ is 2, we get that $au+bv$ and $c'u+d'v$ are linearly independent over $\mathbb{C}$. This shows that $ad'-bc'\neq 0$, as requested.

(2) Let $c':=\frac{1}{2}(c+\overline{c})$ and $d':=\frac{1}{2}(d+\overline{d})$. We note that $c'u+d'v-\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(cu+dv-\zeta_\Omega(au+bv))$. Therefore, it only remains to show that $(au+bv,c'u+d'v)\in\mathrm{GL}_2(\mathbb{C})$. Since α is an isomorphism, we can take the change of variable $x=au+bv$ and $y=cu+dv$. Let $A,B\in\mathbb{C}$ be such that $c'u+d'v=Ax+By$. So $y-\zeta_\Omega(x)$ is algebraic over $\mathbb{C}(Ax+By-\zeta_\Omega(x))$. Evaluating $x=x_0$ for some $x_0\notin\Omega$, we get that y is algebraic over $\mathbb{C}(By)$. So $B\neq 0$ and, hence, $au+bv$ and $c'u+d'v$ are linearly independent over $\mathbb{C}$. This shows that $ad'-bc'\neq 0$, as requested.

(3) Let $S:=\{e^{(a+\overline{a})u+(b+\overline{b})v},e^{(a-\overline{a})u+(b-\overline{b})v}\}$. We note that the elements of S are algebraic over $\mathbb{C}(e^{au+bv})$. Furthermore, e^{au+bv} is algebraic over

$\mathbb{C}(s \mid s \in S)$, so there exist $\delta \in \{1, i\}$ and $a', b' \in \mathbb{R}$ such that e^{au+bv} is algebraic over $\mathbb{C}(e^{\delta(a'u+b'v)})$.

We now show that $(a'u + b'v, cu + dv) \in \mathrm{GL}_2(\mathbb{C})$. It is enough to show that $au+bv$ and $a'u+b'v$ are linearly dependent over $\mathbb{C}$, because then $a'u+b'v$ and $cu + dv$ are linearly independent over $\mathbb{C}$. Suppose by a contradiction that $au+bv$ and $a'u+b'v$ are linearly independent over $\mathbb{C}$. Then, we take the change of variable $x = au + bv$ and $y = a'u + b'v$ to get that e^x is algebraic over $\mathbb{C}(e^y)$, which is clearly a contradiction.

(4) Define

$$S := \{e^{(a+\overline{a})u+(b+\overline{b})v}, e^{(a-\overline{a})u+(b-\overline{b})v}, e^{(c+\overline{c})u+(d+\overline{d})v}, e^{(c-\overline{c})u+(d-\overline{d})v}\}$$

We note that the elements of S are algebraic over $\mathbb{C}(e^{au+bv}, e^{cu+dv})$. Furthermore, (e^{au+bv}, e^{cu+dv}) is algebraic over $\mathbb{C}(s \mid s \in S)$, so we can pick $s_1, s_2 \in S$ such that s_1 and s_2 are algebraically independent over $\mathbb{C}$. We note that, for each $s \in S$, there exist $a', b' \in \mathbb{R}$ and $\delta \in \{1, i\}$ such that $s = e^{\delta(a'u+b'v)}$. Thus, there exist $\delta_1, \delta_2 \in \{1, i\}$ and $a', b', c', d' \in \mathbb{R}$ such that (e^{au+bv}, e^{cu+dv}) is algebraic over $\mathbb{C}(e^{\delta_1(a'u+b'v)}, e^{\delta_2(c'u+d'v)})$.

We now show that $(a'u + b'v, c'u + d'v) \in \mathrm{GL}_2(\mathbb{R})$. Suppose by contradiction that $a'd' - b'c' = 0$. Then, there exists $\lambda \in \mathbb{C}^*$ such that $c'u + d'v = \lambda(a'u + b'v)$. On the other hand, either $au + bv$ or either $cu + dv$ must be linearly independent to $a'u + b'v$, suppose that it is for instance $au + bv$. If we perform the change of variable $x = au + bv$ and $y = a'u + b'v$, we get that e^x is algebraic over $\mathbb{C}(e^y, e^{\lambda y})$, which is clearly a contradiction.

(5) Since we cannot have $c = d = 0$, we may assume that $c \neq 0$. Suppose first that $c + \overline{c} \neq 0$. Hence, $e^{(c+\overline{c})u+(d+\overline{d})v}\widetilde{\sigma}_{\Omega,\xi}(au + bv)\widetilde{\sigma}_{\Omega,\overline{\xi}}(au + bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$. We recall that, by [**11**, Ch.IV, Theorem 4], the function $\widetilde{\sigma}_{\Omega,\xi}(u) \cdot \widetilde{\sigma}_{\Omega,\overline{\xi}}(u) \cdot \widetilde{\sigma}_{\Omega,-\xi-\overline{\xi}}(u)$ is elliptic of period Ω and, therefore, algebraic over $\mathbb{C}(\wp_\Omega(u))$. So, $e^{-(c+\overline{c})u-(d+\overline{d})v}\widetilde{\sigma}_{\Omega,-\xi-\overline{\xi}}(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$ and therefore $(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au + bv))$ is algebraic over

$$\mathbb{C}(\wp_\Omega(au + bv), e^{-(c+\overline{c})u-(d+\overline{d})v}\widetilde{\sigma}_{\Omega,-\xi-\overline{\xi}}(au + bv)).$$

Hence, we can take $\delta := 1$, $c' := -(c + \overline{c})$, $d' := -(d + \overline{d})$ and $\xi' := -\xi - \overline{\xi}$.

Suppose now that $c+\overline{c} = 0$. Then, $c-\overline{c} \neq 0$. By Lemma 4.21.(4), we have that $e^{-(\overline{c}u+\overline{d}v)}\widetilde{\sigma}_{\Omega,-\overline{\xi}}(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$. Hence, $e^{(c-\overline{c})u+(d-\overline{d})v}\widetilde{\sigma}_{\Omega,\xi}(au + bv)\widetilde{\sigma}_{\Omega,-\overline{\xi}}(au + bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au + bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au + bv))$. Reasoning as in the previous case, we get that $(\wp_\Omega(au + bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au + bv))$ is algebraic over

$$\mathbb{C}(\wp_\Omega(au + bv), e^{-(c-\overline{c})u-(d-\overline{d})v}\widetilde{\sigma}_{\Omega,-\xi+\overline{\xi}}(au + bv)).$$

So we can take $\delta := i$, $c' := i(c - \overline{c})$, $d' := i(d - \overline{d})$ and $\xi' := -i(-\xi + \overline{\xi})$.

We now show that $(au + bv, c'u + d'v) \in \mathrm{GL}_2(\mathbb{C})$. Suppose by contradiction that $au + bv$ and $c'u + d'v$ are linearly dependent over $\mathbb{C}$. Then, there exists $\lambda \in \mathbb{C}^*$ such that $c'u + d'v = \lambda(au + bv)$. If we perform the change of variable $x = au + bv$ and $y = cu + dv$ then $e^{\delta\lambda x}\widetilde{\sigma}_{\Omega,\delta\xi'}(x)$ is algebraic over $\mathbb{C}(\wp_\Omega(x), e^y\widetilde{\sigma}_{\Omega,\xi}(x))$. Thus, $e^y\widetilde{\sigma}_{\Omega,\xi}(x)$ is algebraic over

$\mathbb{C}(\wp_\Omega(x), e^{\delta\lambda x}\widetilde{\sigma}_{\Omega,\delta\xi'}(x))$. Evaluating $x = x_0$, for some $x_0 \notin \Omega$, we achieve a contradiction. □

We are now ready to give the classification of the abelian two-dimensional locally Nash groups. If $\omega \in \mathbb{C}\setminus\mathbb{R}$, we denote by $\wp_\omega$, ζ_ω, σ_ω, $\widetilde{\sigma}_{\omega,\xi}$ the functions $\wp_{\langle 1,\omega\rangle_\mathbb{Z}}$ and $\zeta_{\langle 1,\omega\rangle_\mathbb{Z}}$, $\sigma_{\langle 1,\omega\rangle_\mathbb{Z}}$ and $\widetilde{\sigma}_{\langle 1,\omega\rangle_\mathbb{Z},\xi}$ respectively.

Theorem 5.10. (I) *Every two-dimensional simply connected abelian locally Nash group is isomorphic to a group of one and only one of the following types:*

(1) *A direct product of one-dimensional locally Nash groups with charts* id, exp, sin *or* $\wp_{ai}$ *for some* $a \in \mathbb{R}^*$.
(2) $(\mathbb{R}^2, +, (\wp_{ai}(u), v - \zeta_{ai}(u)))$, *for some* $a \in \mathbb{R}^*$.
(3) $(\mathbb{R}^2, +, (\wp_{ai}(u), e^v\widetilde{\sigma}_{ai,\xi}(u)))$, *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R}\setminus\mathbb{Q}$.
(4) $(\mathbb{R}^2, +, (\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv}\widetilde{\sigma}_{ai,-\xi i}(u))))$, *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R}\setminus a\mathbb{Q}$.
(5) *The universal covering group* $(\mathbb{R}^2, +, f)$ *of the connected component of the real points of a simple abelian surface defined over* $\mathbb{R}$.

(II) *The isomorphism classes within each type are defined as follows:*

(i) *Two groups of type* (1) *are isomorphic if and only if their factor groups are isomorphic, where* $(\mathbb{R}, +, \wp_{ai})$ *and* $(\mathbb{R}, +, \wp_{bi})$ *are isomorphic if and only if* $a/b \in \mathbb{Q}^*$.
(ii) *Two groups of type* (2), *defined by* a *and* b *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}^*$.
(iii) *Two groups of type* (3), *defined by* (a, ξ_1) *and* (b, ξ_2) *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}$ *and* $\xi_2 \in \mathbb{Q} + \xi_1\mathbb{Q}^*$.
(iv) *Two groups of type* (4), *defined by* (a, ξ_1) *and* (b, ξ_2) *respectively, are isomorphic if and only if* $a/b \in \mathbb{Q}$ *and* $\xi_2 \in a\mathbb{Q} + \xi_1\mathbb{Q}^*$.
(v) *Two groups of type* (5) *are isomorphic if and only if there is an isogeny between the corresponding abelian surfaces.*

Related to the isomorphism classes of groups of type (5), we remark that, given a meromorphic map $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ admitting an AAT, it may happen that $(\mathbb{R}^2, +, f)$ is the universal covering of a real abelian surface that is not a direct product of two real elliptic curves whereas $(\mathbb{C}^2, +, f)$ is locally $\mathbb{C}$-Nash isomorphic to the universal covering of a direct product of two elliptic curves (see [**22**, Example 48]).

Proof of Theorem 5.10. Along the proof we will use Lemma 3.9 without mention.

(I) By Theorem 3.8, every simply connected two-dimensional locally Nash group is isomorphic to some group $(\mathbb{R}^2, +, f)$, where $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ is an invariant meromorphic map admitting an AAT. By Remark 3.14, $(\mathbb{C}^2, +, f)$ is a locally $\mathbb{C}$-Nash group and, in particular, f admits an AAT and so it is under the hypothesis of Fact 4.10. Hence, there exist $i \in \{1, \dots, 6\}$ and $\mathbb{L} \in \mathcal{P}_i$ such that f is algebraic over $\mathbb{L}$.

[*Case* $\mathbb{L} \in \mathcal{P}_1$] In this case, there exists $\alpha(u, v) = (au + bv, cu + dv) \in \mathrm{GL}_2(\mathbb{C})$ such that $f(u, v)$ is algebraic over $\mathbb{C}(au + bv, cu + dv)$. So $f(u, v)$ is algebraic over $\mathbb{C}(u, v)$ and hence over $\mathbb{R}(u, v)$. Therefore, the identity map is an isomorphism from $(\mathbb{R}^2, +, \mathrm{id}\times\mathrm{id})$ to $(\mathbb{R}^2, +, f)$.

[*Case* $\mathbb{L} \in \mathcal{P}_2$] In this case, there exists $\alpha(u,v) = (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$ such that $f(u,v)$ is algebraic over $\mathbb{C}(e^{au+bv}, cu+dv)$. Since f is an invariant meromorphic function, by Lemma 3.3 $(e^{\bar{a}u+\bar{b}v}, \bar{c}u+\bar{d}v)$ is algebraic over $\mathbb{C}(e^{au+bv}, cu+dv)$.

We prove that $e^{\bar{a}u+\bar{b}v}$ is algebraic over $\mathbb{C}(e^{au+bv})$. Otherwise, $cu+dv$ is algebraic over $\mathbb{C}(e^{au+bv}, e^{\bar{a}u+\bar{b}v})$. By changing (u,v) with (v,u) if necessary, we may assume that $c \neq 0$. Evaluating $v=0$, we get that u is algebraic over $\mathbb{C}(e^{au}, e^{\bar{a}u})$, which is clearly a contradiction (see Fact 4.17).

We also note that $\bar{c}u+\bar{d}v$ is algebraic over $\mathbb{C}(cu+dv)$. Otherwise, e^{au+bv} is algebraic over $\mathbb{C}(cu+dv, \bar{c}u+\bar{d}v)$. Evaluating $v=0$, we get that e^{au} is algebraic over $\mathbb{C}(u)$, which is a contradiction.

We now show that there exists $\gamma \in \mathrm{GL}_2(\mathbb{R})$ and $\delta \in \{1, i\}$ such that $f(\gamma(u,v))$ is algebraic over $\mathbb{C}(e^{\delta u}, v)$. By Lemma 5.9.(1), there exist $c', d' \in \mathbb{R}$ such that $cu+dv$ is algebraic over $\mathbb{C}(c'u+d'v)$ and $(au+bv, c'u+d'v) \in \mathrm{GL}_2(\mathbb{C})$. By Lemma 5.9.(3), there exist $a', b' \in \mathbb{R}$ and $\delta \in \{1, i\}$ such that e^{au+bv} is algebraic over $\mathbb{C}(e^{\delta(a'u+b'v)})$ and $(a'u+b'v, c'u+d'v) \in \mathrm{GL}_2(\mathbb{R})$. Since $f(u,v)$ is algebraic over $\mathbb{C}(e^{au+bv}, cu+dv)$, we deduce that $f(u,v)$ is algebraic over $\mathbb{C}(e^{\delta(a'u+b'v)}, c'u+d'v)$. Now, the existence of γ is clear.

We distinguish two cases:

Subcase $\delta = 1$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^u, v)$.

Subcase $\delta = i$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^{iu}, v)$ and, hence, over $\mathbb{R}(\sin(u), v)$.

[*Case* $\mathbb{L} \in \mathcal{P}_3$] In this case, there is $\alpha(u,v) = (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$ such that $f(u,v)$ is algebraic over $\mathbb{C}(e^{au+bv}, e^{cu+dv})$. By Lemma 3.3, since f is an invariant meromorphic function, $(e^{\bar{a}u+\bar{b}v}, e^{\bar{c}u+\bar{d}v})$ is algebraic over $\mathbb{C}(e^{au+bv}, e^{cu+dv})$. By Lemma 5.9.(4), there exist $a', b', c', d' \in \mathbb{R}$ and $\delta_1, \delta_2 \in \{1, i\}$ such that (e^{au+bv}, e^{cu+dv}) is algebraic over $\mathbb{C}(e^{\delta_1(a'u+b'v)}, e^{\delta_2(c'u+d'v)})$ and $(a'u+b'v, c'u+d'v) \in \mathrm{GL}_2(\mathbb{R})$. If we take $\gamma^{-1}(u,v) := (a'u+b'v, c'u+d'v)$, we get that $f(\gamma(u,v))$ is algebraic over $\mathbb{C}(e^{\delta_1 u}, e^{\delta_2 v})$. We distinguish the following cases.

Subcase $\delta_1 = 1$ *and* $\delta_2 = 1$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^u, e^v)$.

Subcase $\delta_1 = 1$ *and* $\delta_2 = i$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^u, e^{iv})$, so it is algebraic over $\mathbb{R}(e^u, \sin(v))$. Therefore, we get that γ is an isomorphism from $(\mathbb{R}^2, +, (e^u, \sin(v)))$ to $(\mathbb{R}^2, +, f(u,v))$.

Subcase $\delta_1 = i$ *and* $\delta_2 = 1$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^{iu}, e^v)$. Permuting the variables u and v, we are as in the subcase $\delta_1 = 1$ and $\delta_2 = i$.

Subcase $\delta_1 = i$ *and* $\delta_2 = i$. Then, $f(\gamma(u,v))$ is algebraic over $\mathbb{R}(e^{iu}, e^{iv})$, so it is algebraic over $\mathbb{R}(\sin(u), \sin(v))$.

[*Case* $\mathbb{L} \in \mathcal{P}_4$] We need some preliminaries. Given a lattice $\Omega < (\mathbb{C}, +)$ and $\xi \in \{0, 1\}$, we denote by $g_{4,\xi,\Omega}(u,v)$ the function $(\wp_\Omega(u), v - \xi\zeta_\Omega(u))$. Also, given $\tau \in \mathbb{C} \setminus \mathbb{R}$, we denote by $g_{4,\xi,\tau}$ the function $g_{4,\xi,\langle 1,\tau\rangle_{\mathbb{Z}}}$. We first prove the following claim.

Claim 1*: Let* $\alpha(u,v) = (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$ *with* $a \neq 0$, $\xi \in \{0,1\}$ *and* Ω *a lattice of* $(\mathbb{C}, +)$. *Suppose that an invariant meromorphic function* $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ *is algebraic over* $\mathbb{C}(g_{4,\xi,\Omega} \circ \alpha)$*:*

(1) *Then* $a^{-1}\Omega$ *contains an invariant sublattice.*

(2) *If in addition* $a \in \mathbb{R}$ *and* Ω *is an invariant lattice then there exists* $\gamma \in \mathrm{GL}_2(\mathbb{R})$ *such that* $f \circ \gamma$ *is algebraic over* $\mathbb{C}(g_{4,\xi,\Omega})$.

Proof of Claim 1. By Lemma 5.1, $\overline{g_{4,\xi,\Omega}(\overline{u},\overline{v})} = g_{4,\xi,\overline{\Omega}}(u,v)$. Since f is an invariant meromorphic function, by Lemmas 3.3 and 3.4 $g_{4,\xi,\overline{\Omega}}(u,v) \circ \overline{\alpha(\overline{u},\overline{v})}$ is algebraic over $\mathbb{C}(g_{4,\xi,\Omega}(u,v) \circ \alpha(u,v))$. So $(\wp_{\overline{\Omega}}(\overline{a}u+\overline{b}v), \overline{c}u+\overline{d}v-\xi\zeta_{\overline{\Omega}}(\overline{a}u+\overline{b}v))$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), cu+dv-\xi\zeta_\Omega(au+bv))$.

We first prove by contradiction that there exists $\lambda \in \mathbb{C}^*$ such that $\lambda(\overline{a}u+\overline{b}v) = au+bv$. Otherwise, $\overline{a}u+\overline{b}v$ and $au+bv$ are linearly independent over $\mathbb{C}$. If we perform the change of variable $x = au+bv$ and $y = \overline{a}u+\overline{b}v$, we get that $cu+dv = Ax+By$ for some unique $A, B \in \mathbb{C}$. By hypothesis, $\wp_{\overline{\Omega}}(y)$ is algebraic over $\mathbb{C}(\wp_\Omega(x), Ax+By-\xi\zeta_\Omega(x))$. Evaluating $x = x_0$, for some $x_0 \notin \Omega$, we deduce $\wp_{\overline{\Omega}}(y)$ is algebraic over $\mathbb{C}(By)$, which contradicts Fact 4.17.

Now, we show (1). We prove by contradiction that $\wp_{\overline{\Omega}}(\overline{a}u+\overline{b}v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv))$. Otherwise, $cu+dv-\xi\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), \wp_{\overline{\Omega}}(\overline{a}u+\overline{b}v))$. Consider the change of variable $x = au+bv$ and $y = cu+dv$ and recall that there exists $\lambda \in \mathbb{C}^*$ such that $\lambda(\overline{a}u+\overline{b}v) = au+bv$. Then, $y-\xi\zeta_\Omega(x)$ is algebraic over $\mathbb{C}(\wp_\Omega(x), \wp_{\overline{\Omega}}(\lambda x))$. Evaluating at $x = x_0$, for some $x_0 \notin \Omega$, we achieve the desired contradiction. Next, by Fact 4.14, we know that $\wp_{\overline{a^{-1}\Omega}}(u+\overline{a}^{-1}\overline{b}v)$ is algebraic over $\mathbb{C}(\wp_{a^{-1}\Omega}(u+a^{-1}bv))$. If we evaluate $v = 0$, we get that $\wp_{\overline{a^{-1}\Omega}}(u)$ is algebraic over $\mathbb{C}(\wp_{a^{-1}\Omega}(u))$ and, hence, we are under the hypothesis of Lemma 5.3.

We show (2). Note first that, since $a \in \mathbb{R}$, we must have $\lambda = 1$ and, therefore, $b \in \mathbb{R}$. In particular, $\overline{c}u+\overline{d}v-\xi\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), cu+dv-\xi\zeta_\Omega(au+bv))$. Moreover, we show that $\overline{c}u+\overline{d}v-\xi\zeta_\Omega(au+bv)$ is algebraic over $\mathbb{C}(cu+dv-\xi\zeta_\Omega(au+bv))$. Otherwise, $\wp_\Omega(au+bv)$ is algebraic over $\mathbb{C}(cu+dv-\xi\zeta_\Omega(au+bv), \overline{c}u+\overline{d}v-\xi\zeta_\Omega(au+bv))$. Evaluating $v = 0$, we deduce that $\wp_\Omega(au)$ is algebraic over $\mathbb{C}(u, \xi\zeta_\Omega(au))$, which contradicts Fact 4.14. Finally, by Lemma 5.9.(1) and (2) and since $a, b \in \mathbb{R}$, there exist $c', d' \in \mathbb{R}$ such that $\mathbb{C}(cu+dv)$ is algebraic over $\mathbb{C}(c'u+d'v)$ and $(au+bv, c'u+d'v) \in \mathrm{GL}_2(\mathbb{R})$. Since $f(u,v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), cu+dv-\xi\zeta_\Omega(au+bv))$, we get that $f(u,v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), c'u+d'v-\xi\zeta_\Omega(au+bv))$. So we can take $\gamma^{-1}(u,v) := (au+bv, c'u+d'v)$. This ends the proof of Claim 1. □

Since we are in case $\mathbb{L} \in \mathcal{P}_4$, f must be algebraic over $\mathbb{C}(g_{4,\xi,\Omega} \circ \alpha)$, for some $\alpha := (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$, $\xi \in \{0,1\}$ and $\Omega < (\mathbb{C},+)$. We can assume that $a \neq 0$. For, $(\mathbb{R}^2,+,f(u,v))$ is isomorphic to $(\mathbb{R}^2,+,f(v,u))$ via the linear map $(u,v) \mapsto (v,u)$ and, therefore, we can replace without loss of generality $f(u,v)$ and $\alpha(u,v)$ by $f(v,u)$ and $\alpha(v,u)$, respectively. In particular, by Claim 1.(1) and Remark 5.4, there exist $A, B \in \mathbb{R}$ such that $\Omega_1 := \langle A, Bi\rangle_\mathbb{Z}$ is a sublattice of $a^{-1}\Omega$.

Subcase $\xi = 0$. Let $\beta(u,v) := (u+a^{-1}bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$. By Fact 4.14.(2), $g_{4,0,\Omega} \circ \alpha$ is algebraic over $\mathbb{C}(g_{4,0,a^{-1}\Omega} \circ \beta)$. On the other hand, by Lemma 4.6.(1), we get that $g_{4,0,a^{-1}\Omega}$ is algebraic over $\mathbb{C}(g_{4,0,\Omega_1})$. Let $\beta_1(u,v) := (Au, v) \in \mathrm{GL}_2(\mathbb{C})$ and $\tau := A^{-1}Bi$. By Fact 4.14.(2), the map $g_{4,0,\Omega_1} \circ \beta_1$ is algebraic over $\mathbb{C}(g_{4,0,\tau})$. Let $\alpha_1 := \beta_1^{-1} \circ \beta \in \mathrm{GL}_2(\mathbb{C})$, where $\alpha_1(u,v) = (A^{-1}u+A^{-1}a^{-1}bv, cu+dv)$. We note that f is algebraic over

$\mathbb{C}(g_{4,0,\tau} \circ \alpha_1)$. By Claim 1.(2), there exists $\gamma \in \mathrm{GL}_2(\mathbb{R})$ such that $f \circ \gamma$ is algebraic over $\mathbb{C}(g_{4,0,\tau})$ and, hence, over $\mathbb{R}(g_{4,0,\tau})$.

Subcase $\xi = 1$. Let $\beta(u,v) := (u + a^{-1}bv, a^{-1}cu + a^{-1}dv) \in \mathrm{GL}_2(\mathbb{C})$. By the proof of Lemma 4.15, we get that $g_{4,1,\Omega} \circ \alpha$ is algebraic over $\mathbb{C}(g_{4,1,a^{-1}\Omega} \circ \beta)$. Let $\beta_1 := (u, \mathfrak{c}(a^{-1}\Omega, \Omega_1)u + [a^{-1}\Omega : \Omega_1]v) \in \mathrm{GL}_2(\mathbb{C})$. By Lemma 4.16, $g_{4,1,a^{-1}\Omega} \circ \beta_1$ is algebraic over $\mathbb{C}(g_{4,1,\Omega_1})$. Let $\beta_2 := (Au, A^{-1}v) \in \mathrm{GL}_2(\mathbb{C})$ and $\tau := A^{-1}Bi$. By Lemma 4.15, we get that $g_{4,1,\Omega_1} \circ \beta_2$ is algebraic over $\mathbb{C}(g_{4,1,\tau})$. Let $\alpha_1 := \beta_2^{-1} \circ \beta_1^{-1} \circ \beta \in \mathrm{GL}_2(\mathbb{C})$, where $\alpha_1(u,v) = (a_1u + b_1v, c_1u + d_1v)$. We note that $a_1 = A^{-1} \in \mathbb{R}$ and f is algebraic over $\mathbb{C}(g_{4,1,\tau} \circ \alpha_1)$. By Claim 1.(2), there exists $\gamma \in \mathrm{GL}_2(\mathbb{R})$ such that $f \circ \gamma$ is algebraic over $\mathbb{C}(g_{4,1,\tau})$ and, hence, over $\mathbb{R}(g_{4,1,\tau})$.

[*Case* $\mathbb{L} \in \mathcal{P}_5$] We need some preliminaries. Given a lattice $\Omega < (\mathbb{C},+)$ and $\xi \in \mathbb{C}$, we denote $g_{5,\xi,\Omega}(u,v)$ the function $(\wp_\Omega(u), e^v\widetilde{\sigma}_{\Omega,\xi}(u))$, where $\widetilde{\sigma}_{\Omega,\xi}(u) = \sigma_\Omega(u-\xi)/\sigma_\Omega(u)$. Also, given $\tau \in \mathbb{C} \setminus \mathbb{R}$, we denote $g_{5,\xi,\tau}$ the function $g_{5,\xi,\langle 1,\tau\rangle_{\mathbb{Z}}}$. We first prove the following claim.

Claim 2: *Let* $\alpha(u,v) = (au+bv, cu+dv) \in \mathrm{GL}_2(\mathbb{C})$ *with* $a \neq 0$*,* $\xi \in \mathbb{C}$ *and* Ω *a lattice of* $(\mathbb{C},+)$*. If an invariant meromorphic function* $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ *is algebraic over* $\mathbb{C}(g_{5,\xi,\Omega} \circ \alpha)$*:*

(1) *Then* $a^{-1}\Omega$ *contains an invariant sublattice.*
(2) *If in addition* $a \in \mathbb{R}$ *and* Ω *is an invariant lattice then there exists* $\gamma \in \mathrm{GL}_2(\mathbb{R})$*,* $\xi' \in \mathbb{R}$ *and* $\delta \in \{1, i\}$ *such that* $f \circ \gamma$ *is algebraic over* $\mathbb{C}(\wp_\Omega(u), e^{\delta v}\widetilde{\sigma}_{\Omega,\delta\xi'}(u))$.

Proof of Claim 2. By Lemma 5.1, $\overline{g_{5,\xi,\tau}(\overline{u},\overline{v})} = g_{5,\overline{\xi},\overline{\tau}}(u,v)$. Since f is an invariant meromorphic function, by Lemmas 3.3 and 3.4 we get that $g_{5,\overline{\xi},\overline{\Omega}}(u,v) \circ \overline{\alpha(\overline{u},\overline{v})}$ is algebraic over $\mathbb{C}(g_{5,\xi,\Omega}(u,v) \circ \alpha(u,v))$. So $(\wp_{\overline{\Omega}}(\overline{a}u + \overline{b}v), e^{\overline{c}u+\overline{d}v}\widetilde{\sigma}_{\overline{\Omega},\overline{\xi}}(\overline{a}u+\overline{b}v))$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$.

We first prove that there exists $\lambda \in \mathbb{C}^*$ such that $\overline{a}u + \overline{b}v = \lambda(au+bv)$. Otherwise, $\overline{a}u + \overline{b}v$ and $au+bv$ are linearly independent over $\mathbb{C}$. If we perform the change of variable $x = au+bv$ and $y = \overline{a}u + \overline{b}v$, we get that $cu + dv = Ax + By$ for some unique $A, B \in \mathbb{C}$. By hypothesis, $\wp_{\overline{\Omega}}(y)$ is algebraic over $\mathbb{C}(\wp_\Omega(x), e^{Ax+By}\widetilde{\sigma}_{\Omega,\xi}(x))$. If we evaluate $x = x_0$, for some $x_0 \notin \Omega$, we get that $\wp_{\overline{\Omega}}(y)$ is algebraic over $\mathbb{C}(e^{By})$, which contradicts Fact 4.17.

Now, we show (1). It is enough to proof that $\wp_{\overline{\Omega}}(\overline{a}u + \overline{b}v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv))$, since then the same proof of Claim 1.(1) shows that $a^{-1}\Omega \cap \overline{a^{-1}\Omega}$ is an invariant sublattice of $a^{-1}\Omega$. Suppose by contradiction that $\wp_{\overline{\Omega}}(\overline{a}u+\overline{b}v)$ is not algebraic over $\mathbb{C}(\wp_\Omega(au+bv))$. Then, $e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), \wp_{\overline{\Omega}}(\overline{a}u+\overline{b}v))$. Consider the change of variable $x = au+bv$ and $y = cu+dv$. Then, $e^y\widetilde{\sigma}_{\Omega,\xi}(x)$ is algebraic over $\mathbb{C}(\wp_\Omega(x), \wp_{\overline{\Omega}}(\lambda x))$. This is clearly a contradiction, because the latter functions do not depend on y.

We show (2). Note first that, since $a \in \mathbb{R}$, we must have $\lambda = 1$ and therefore $b \in \mathbb{R}$. In particular, $e^{\overline{c}u+\overline{d}v}\widetilde{\sigma}_{\Omega,\overline{\xi}}(au+bv)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$. By Lemma 5.9.(5) and since $a, b \in \mathbb{R}$, there exist

$c', d', \xi' \in \mathbb{R}$ and $\delta \in \{1, i\}$ such that $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{\delta(c'u+d'v)}\widetilde{\sigma}_{\Omega,\delta\xi'}(au+bv)$ and $(au+bv, c'u+d'v) \in \mathrm{GL}_2(\mathbb{R})$. Since $f(u,v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{cu+dv}\widetilde{\sigma}_{\Omega,\xi}(au+bv))$, we get that $f(u,v)$ is algebraic over $\mathbb{C}(\wp_\Omega(au+bv), e^{c'u+d'v}\widetilde{\sigma}_{\Omega,\delta\xi'}(au+bv))$. So we can take $\gamma^{-1}(u,v) := (au+bv, c'u+d'v)$. This ends the proof of Claim 2. $\square$

Since we are in the case $\mathbb{L} \in \mathcal{P}_5$, f must be algebraic over $\mathbb{C}(g_{5,\xi,\Omega} \circ \alpha)$, for some $\alpha \in \mathrm{GL}_2(\mathbb{C})$, $\xi \in \mathbb{C}$ and $\Omega < (\mathbb{C}, +)$. As in the case $\mathbb{L} \in \mathcal{P}_4$, we can assume $a \neq 0$. By Claim 2.(1) and Remark 5.4, there exist $A, B \in \mathbb{R}$ such that $\Omega_1 := \langle A, Bi\rangle_{\mathbb{Z}}$ is a sublattice of $a^{-1}\Omega$. Let $\beta(u,v) := (u + a^{-1}bv, cu + dv) \in \mathrm{GL}_2(\mathbb{C})$. By the proof of Lemma 4.19, we get that $g_{5,\xi,\Omega}\circ\alpha$ is algebraic over $\mathbb{C}(g_{5,a^{-1}\xi,a^{-1}\Omega} \circ \beta)$. Let $\beta_1 := (u, a^{-1}\xi\mathfrak{c}(a^{-1}\Omega, \Omega_1)u + [a^{-1}\Omega : \Omega_1]v) \in \mathrm{GL}_2(\mathbb{C})$. By Lemma 4.20, $g_{5,a^{-1}\xi,a^{-1}\Omega} \circ \beta_1$ is algebraic over $\mathbb{C}(g_{5,a^{-1}\xi,\Omega_1})$. Let $\beta_2 := (Au, v) \in \mathrm{GL}_2(\mathbb{C})$ and $\tau := A^{-1}Bi$. By Lemma 4.19, the map $g_{5,a^{-1}\xi,\Omega_1} \circ \beta_2$ is algebraic over $\mathbb{C}(g_{5,A^{-1}a^{-1}\xi,\tau})$. Let $\alpha_1 := \beta_2^{-1} \circ \beta_1^{-1} \circ \beta \in \mathrm{GL}_2(\mathbb{C})$, $\alpha_1(u,v) = (a_1u + b_1v, c_1u + d_1v)$. We note that $a_1 = A^{-1} \in \mathbb{R}$ and f is algebraic over $\mathbb{C}(g_{5,A^{-1}a^{-1}\xi,\tau} \circ \alpha_1)$. By Claim 2.(2), there exist $\gamma \in \mathrm{GL}_2(\mathbb{R})$, $\xi' \in \mathbb{R}$ and $\delta \in \{1, i\}$ such that $f(\gamma(u,v))$ is algebraic over $\mathbb{C}(\wp_\Omega(u), e^{\delta v}\widetilde{\sigma}_{\Omega,\delta\xi'}(u))$. We distinguish two subcases.

Subcase $\delta = 1$. Then, $f \circ \gamma$ is algebraic over $\mathbb{R}(\wp_\tau(u), e^v\widetilde{\sigma}_{\tau,\xi'}(u))$. We will show in (II) that $(\wp_\tau(u), e^v\widetilde{\sigma}_{\tau,\xi'}(u))$ is algebraic over $\mathbb{R}((\wp_\tau(u), e^v))$ if and only if $\xi' \in \mathbb{Q}$.

Subcase $\delta = i$. Then, $f \circ \gamma$ is algebraic over $\mathbb{R}(\wp_\tau(u), e^{iv}\widetilde{\sigma}_{\tau,i\xi'}(u))$. Since $f \circ \gamma$ is an invariant meromorphic function, by Lemma 3.3.(3) the imaginary part of $e^{iv}\widetilde{\sigma}_{\tau,i\xi'}(u)$,

$$\frac{1}{2i}(e^{iv}\widetilde{\sigma}_{\tau,i\xi'}(u) - e^{-iv}\widetilde{\sigma}_{\tau,-i\xi'}(u))$$

is algebraic over $\mathbb{R}(\wp_\tau(u), e^{iv}\widetilde{\sigma}_{\tau,i\xi'}(u))$. Moreover, this imaginary part is algebraically independent of $\wp_\tau(u)$, because the latter do not depend on v. Therefore, γ is an isomorphism from $(\mathbb{R}^2, +, f(u,v))$ to

$$(\mathbb{R}^2, +, (\wp_\tau(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{\tau,i\xi'}(u) - e^{-iv}\widetilde{\sigma}_{\tau,-i\xi'}(u)))).$$

We remark that the above group is isomorphic to $(\mathbb{R}^2, +, (\wp_\tau(u), \sin(v)))$ if and only if $\xi' \in \tau\mathbb{Q}$. This will be proved in (II).

[*Case* $\mathbb{L} \in \mathcal{P}_6$] In this case the coordinates functions of $f := (f_1, f_2)$ are algebraic over a field of the family $\mathcal{P}_6$, *i.e.*, over a field of the form $\mathbb{C}(\Lambda)$, for some lattice $\Lambda < (\mathbb{C}^2, +)$, and satisfying $\operatorname{tr.deg.}_{\mathbb{C}} \mathbb{C}(\Lambda) = 2$. In particular, by Lemmas 3.13.(5) and 3.15, the period group Λ_f of f is also a lattice of $(\mathbb{C}^2, +)$ and, since $f_1, f_2 \in \mathbb{C}(\Lambda_f)$ are algebraically independent, $\operatorname{tr.deg.}_{\mathbb{C}} \mathbb{C}(\Lambda_f) = 2$ (see [**45**][Ch. 5 §11 Theorems 5 and 6]).

On the other hand, by the proof of Theorem 4.28.(I), subcase (9), the quotient map $\Psi : (\mathbb{C}^2, +, f) \to \mathbb{C}^2/\Lambda_f$ is a locally $\mathbb{C}$-Nash universal covering map, where the abelian variety $\mathbb{C}^2/\Lambda_f$ is endowed with the canonical structure of complex projective variety. Moreover, since Λ_f is invariant by Lemma 3.13.(5), the conjugation map induces an anti-holomorphic involution σ of $\mathbb{C}^2/\Lambda_f$, so we can assume that $\mathbb{C}^2/\Lambda_f$ is defined over $\mathbb{R}$ as projective algebraic variety (see [**41**, pag. 56]). Note also that $\Psi(\mathbb{R}^2)$ is contained in the fixed points of σ, *i.e.*, the real points of the real abelian variety $\mathbb{C}^2/\Lambda_f$. All in all, $\Psi|_{\mathbb{R}}$ maps $(\mathbb{R}^2,+,f)$ onto the connected component of the real points of $\mathbb{C}^2/\Lambda_f$ with discrete kernel and, therefore, it is a locally Nash covering.

We now check that two groups of different type are not isomorphic. We note that if $(\mathbb{R}^2,+,f)$ and $(\mathbb{R}^2,+,g)$ are isomorphic as locally Nash groups, for some invariant meromorphic maps $f,g:\mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ admitting an AAT, then $(\mathbb{C}^2,+,f)$ and $(\mathbb{C}^2,+,g)$ are also isomorphic as locally $\mathbb{C}$-Nash groups. We also note that if two groups which are direct products of one-dimensional groups are isomorphic then their factor groups must be isomorphic. This two rules, combined with Theorem 4.28 and Fact 5.6, show that no group can be isomorphic to one of a different type. This ends the proof of (I).

(II) We now check the isomorphism classes. We have already shown that two groups of different type are not isomorphic. If $(\mathbb{R}^2,+,\wp_{ai}\times g)$ and $(\mathbb{R}^2,+,\wp_{bi}\times g)$, with $g=\mathrm{id}$, exp or sin and $a,b\in\mathbb{R}^*$, are isomorphic then there exists an isomorphism from $(\mathbb{R},+,\wp_{ai})$ to $(\mathbb{R},+,\wp_{bi})$, so $a/b\in\mathbb{Q}$. The converse is clear, and the classification of isomorphism classes of groups which are a direct product of one-dimensional ones follows easily from this fact.

We begin with the study of the isomorphisms of the groups of type (2). Take $\omega_1,\omega_2\in i\mathbb{R}^*$. If $\alpha(u,v):=(a_{11}u+a_{12}v,a_{21}u+a_{22}v)\in \mathrm{GL}_2(\mathbb{R})$ is an isomorphism from $(\mathbb{R}^2,+,g_{4,1,\omega_1})$ to $(\mathbb{R}^2,+,g_{4,1,\omega_2})$ then, by the Claim of Lemma 4.18, $\wp_{\omega_2}(a_{11}u)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$. Thus, by Fact 5.6, we infer that $\omega_1/\omega_2\in\mathbb{Q}$. Conversely, if $\omega_1/\omega_2\in\mathbb{Q}$ then, by Lemma 4.18 and by Lemma 5.8, there exists $\alpha\in\mathrm{GL}_2(\mathbb{R})$ such that $g_{4,1,\omega_2}\circ\alpha$ is algebraic over $\mathbb{C}(g_{4,1,\omega_1})$.

We study now the isomorphisms between the groups of type (3). We recall that $K_\omega=\mathbb{Q}(\omega)$ if ω is quadratic over $\mathbb{Q}$ and $K_\omega=\mathbb{Q}$ otherwise. Take $\omega_1,\omega_2\in i\mathbb{R}^*$ and $\xi_1,\xi_2\in\mathbb{R}$. If $\alpha(u,v):=(a_{11}u+a_{12}v,a_{21}u+a_{22}v)\in\mathrm{GL}_2(\mathbb{R})$ is an isomorphism from $(\mathbb{R}^2,+,g_{5,\xi_1,\omega_1})$ to $(\mathbb{R}^2,+,g_{5,\xi_2,\omega_2})$ then, by Lemma 4.22, $\wp_{\omega_2}(a_{11}u)$ is algebraic over $\mathbb{C}(\wp_{\omega_1}(u))$. By Fact 5.6, we deduce $\omega_1/\omega_2\in\mathbb{Q}$. By Theorem 4.28.$(II)$.$(ii)$, we also have that $\xi_2\in\langle 1,\omega_1\rangle_{\mathbb{Q}}+\xi_1K^*_{\omega_1}$. Since $\xi_1,\xi_2\in\mathbb{R}$, we get that $\xi_2\in\mathbb{Q}+\xi_1\mathbb{Q}^*$. For the converse, if $\omega_1/\omega_2\in\mathbb{Q}$ and $\xi_2\in\mathbb{Q}+\xi_1\mathbb{Q}^*$ then, by Proposition 4.23 and Lemma 4.24, there exists $\alpha\in\mathrm{GL}_2(\mathbb{C})$ such that $g_{5,\xi_2,\omega_2}\circ\alpha$ is algebraic over $\mathbb{C}(g_{5,\xi_1,\omega_1})$. So we must check that $\alpha\in\mathrm{GL}_2(\mathbb{R})$.

Indeed, following the computations of the above results, if we choose $a,d\in\mathbb{Z}$ such that $\frac{\omega_1}{\omega_2}=\frac{d}{a}$ and we denote by $p,q\in\mathbb{Q}$, $q\neq 0$, the rational numbers satisfying $d\xi_2=p+q\xi_1$, and we take the invariant sublattice $\Omega=\langle d,a\omega_1\rangle_{\mathbb{Z}}$ of $\Omega_1:=\langle 1,\omega_1\rangle_{\mathbb{Z}}$ then

$$\alpha(u,v)=\left(\frac{1}{d}u,-\frac{d\xi_2\mathfrak{c}(\Omega_1,\Omega)+C}{[\Omega_1,\Omega]}u+\frac{q}{[\Omega_1,\Omega]}v\right)\in\mathrm{GL}_2(\mathbb{R}),$$

where $C\in\mathbb{R}$ satisfies $\widetilde{\sigma}_{p_2^{-1}\Omega_1,p}(u)=e^{Cu+D}$ for a certain $D\in\mathbb{R}$ and $p_1,p_2\in\mathbb{Z}$ such that $p=p_1/p_2$. The existence of such C and D was shown in the proof of Lemma 4.21.(5) and we note that both numbers must be real, becayse $\widetilde{\sigma}_{p_2^{-1}\Omega_1,p}$ is invariant.

Next, we study the isomorphisms between the groups of type (4). Let $\omega_1, \omega_2 \in i\mathbb{R}^*$ and $\xi_1, \xi_2 \in \mathbb{R}$. Denote

$$g_{6,i\xi_1,\omega_1}(u,v) := \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{\omega_1,i\xi_1}(u) - e^{-iv}\widetilde{\sigma}_{\omega_1,-i\xi_1}(u)).$$

First note that in the proof of the case $\mathbb{L} \in \mathcal{P}_5$, subcase $\delta = i$, we showed that $g_{6,i\xi_1,\omega_1}$ is algebraic over $\mathbb{C}(g_{5,i\xi,\omega_1} \circ \beta_1)$ where $\beta_1(u,v) = (u, iv)$. Moreover, by Fact 4.14, it is easy to check that for $\gamma_1(u,v) = (\omega_1 u, -iv)$ we have that $g_{5,i\xi,\omega_1} \circ \beta_1 \circ \gamma_1$ is algebraic over $\mathbb{C}(g_{5,\omega_1^{-1} i\xi_1,\omega_1^{-1}})$. Therefore, $g_{6,i\xi_1,\omega_1} \circ \gamma_1$ is algebraic over $\mathbb{C}(g_{5,\omega_1^{-1} i\xi_1,\omega_1^{-1}})$. Similarly, for $\gamma_2(u,v) = (\omega_2 u, -iv)$, so $g_{6,i\xi_2,\omega_2} \circ \gamma_2$ is algebraic over $\mathbb{C}(g_{5,\omega_2^{-1} i\xi_2,\omega_2^{-1}})$. Hence, given $\alpha(u,v) = (au + bv, cu + dv) \in \mathrm{GL}_2(\mathbb{R})$, we deduce that $g_{6,i\xi_2,\omega_2} \circ \alpha$ is algebraic over $\mathbb{C}(g_{6,i\xi_1,\omega_1})$ if and only if $g_{5,\omega_2^{-1} i\xi_2,\omega_2^{-1}} \circ \gamma_2^{-1} \circ \alpha \circ \gamma_1$ is algebraic over $\mathbb{C}(g_{5,\omega_1^{-1} i\xi_1,\omega_1^{-1}})$. Since $(\gamma_2^{-1} \circ \alpha \circ \gamma_1)(u,v) = (\frac{\omega_1}{\omega_2} au - \frac{i}{\omega_2} bv, i\omega_1 cu + dv)$ belongs to $\mathrm{GL}_2(\mathbb{R})$, it follows from case 3 that $g_{5,\omega_2^{-1} i\xi_2,\omega_2^{-1}} \circ \gamma_2^{-1} \circ \alpha \circ \gamma_1$ is algebraic over $\mathbb{C}(g_{5,\omega_1^{-1} i\xi_1,\omega_1^{-1}})$ if and only if $\omega_1/\omega_2 \in \mathbb{Q}^*$ and $\xi_2 \in i\omega_2\mathbb{Q}^* + \frac{\omega_2}{\omega_1}\xi_1\mathbb{Q}^* = i\omega_1\mathbb{Q}^* + \xi_1\mathbb{Q}^*$, as required.

Finally, we study the isomorphisms between the groups of type (5). If $\alpha : (\mathbb{R}^2, +, f) \to (\mathbb{R}^2, +, g)$, where $\alpha \in \mathrm{GL}_2(\mathbb{R})$, is a locally Nash isomorphism between groups of type (5) then $\alpha : (\mathbb{C}^2, +, f) \to (\mathbb{C}^2, +, g)$ is a locally $\mathbb{C}$-Nash isomorphism and, therefore, by the proof of Corollary 2.16, we get an isogeny between $\mathbb{C}^2/\Lambda_f$ and $\mathbb{C}^2/\Lambda_g$ defined over $\mathbb{R}$. Conversely, an isogeny between the real abelian varieties which is defined over $\mathbb{R}$ gives a Nash map between two connected neighborhoods of the identity which is a local isomorphism, which in turn extends to a locally Nash isomorphism between the universal coverings $(\mathbb{R}^2, +, f)$ and $(\mathbb{R}^2, +, g)$. This ends the proof of (II) and, hence, of the theorem. □

Finally, we compute the automorphism groups of the locally Nash groups of Theorem 5.10.

Proposition 5.11. *Let* $(\mathbb{R}^2, +, f)$ *be a locally Nash group.* $\mathrm{Aut}(\mathbb{R}^2, +, f)$ *is one of the following:*

(1) $\mathrm{GL}_2(\mathbb{R})$, *if* $f = \mathrm{id} \times \mathrm{id}$.

(2) $\mathrm{Diag}(\mathbb{Q}^*, \mathbb{R}^*)$, *if* $f = \mathrm{id} \times g$ *with* $g = \exp, \sin$ *or* $\wp_{ai}$, *for some* $a \in \mathbb{R}^*$.

(3) $Gl_2(\mathbb{Q})$ *if* $f = g \times g$, *with* $g = \exp$ *or* $\sin$.

(4) $\mathrm{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$, *if* $f = \exp \times \sin$, $\wp_{ai} \times \exp$ *or* $\wp_{ai} \times \sin$, *for some* $a \in \mathbb{R}^*$.

(5.1) $\mathrm{GL}_2(\mathbb{Q})$, *if* $f = \wp_{ai} \times \wp_{bi}$, *for some* $a, b \in \mathbb{R}^*$ *such that* $a/b \in \mathbb{Q}^*$.

(5.2) $\mathrm{Diag}(\mathbb{Q}^*, \mathbb{Q}^*)$, *if* $f = \wp_{ai} \times \wp_{bi}$, *for some* $a, b \in \mathbb{R}^*$ *such that* $a/b \notin \mathbb{Q}^*$.

(6) $\left\{ q \begin{pmatrix} 1 & 0 \\ \mathfrak{qc}(\Omega, q\Omega) & [\Omega : q\Omega]q^{-2} \end{pmatrix} \,\middle|\, q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$, *if* $f = (\wp_{ai}(u), v - \zeta_{ai}(u))$.

(7) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi\mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \,\middle|\, q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus \mathbb{Q}$, *if* $f = (\wp_{ai}(u), e^v\widetilde{\sigma}_{ai,\xi}(u))$.

(8) $\left\{ q \begin{pmatrix} 1 & 0 \\ \xi \mathfrak{qc}(\Omega, q\Omega) & 1 \end{pmatrix} \;\middle|\; q \in \mathbb{Q}^* \right\}$, *where* $\Omega = \langle 1, ai \rangle_{\mathbb{Z}}$ *for some* $a \in \mathbb{R}^*$ *and* $\xi \in \mathbb{R} \setminus a\mathbb{Q}$, *if* $f = (\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,\xi i}(u) - e^{-iv}\widetilde{\sigma}_{ai,-\xi i}(u)))$.

Proof. Since $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ is an invariant meromorphic map, by Corollary 3.9 we have that α is a locally Nash automorphism of $(\mathbb{R}^2, +, f)$ if and only if $\alpha \in \mathrm{GL}_2(\mathbb{R})$ and α is a locally $\mathbb{C}$-Nash automorphism of $(\mathbb{C}^2, +, f)$. On the other hand, we note that $u \mapsto ui$ is a locally $\mathbb{C}$-Nash isomorphism from $(\mathbb{C}, +, \sin)$ to $(\mathbb{C}, +, \exp)$. Hence, all the locally Nash automorphisms from cases 1 to 7 are an easy consequence of Proposition 4.31 (and Lemma 5.8, in the cases 6 and 7). For example, we explicitely compute the case of $\mathrm{Aut}(\mathbb{R}^2, +, \exp \times \sin)$. The map $\beta(u, v) = (u, iv)$ is a locally $\mathbb{C}$-Nash isomorphism from $(\mathbb{C}^2, +, \exp \times \sin)$ to $(\mathbb{C}^2, +, \exp \times \sin)$. Therefore, $\mathrm{Aut}(\mathbb{C}^2, +, \exp \times \sin) = \beta^{-1} \circ \mathrm{Aut}(\mathbb{C}^2, +, \exp \times \exp) \circ \beta = \{(au + ibv, -icu + dv) \mid a, b, c, d \in \mathbb{Q}, ad - bd \neq 0\}$ and, by Corollary 3.9, we get that $\mathrm{Aut}(\mathbb{R}^2, +, \exp \times \sin) = \{(au, dv) \mid a, d \in \mathbb{Q}^*\}$, as required.

We will also explicitely compute $\mathrm{Aut}(\mathbb{R}^2, +, \wp_{ai} \times \wp_{bi})$. If $a/b \in \mathbb{Q}^*$ then, by Fact 5.5, the identity map is an isomorphism from $(\mathbb{R}, +, \wp_{ai})$ to $(\mathbb{R}, +, \wp_{bi})$ and, therefore, we can follow the proof of case (6.1) of Proposition 4.31 for $\tau = 1$. Otherwise, $(\mathbb{R}, +, \wp_{ai})$ and $(\mathbb{R}, +, \wp_{bi})$ are not isomorphic and, therefore, we can proceed as in case (6.2) of Proposition 4.31.

Finally, we consider case 8. Given $a \in \mathbb{R}^*$ and $\xi \in \mathbb{R} \setminus a\mathbb{Q}$, let

$$g_{6,i\xi,ai}(u, v) := \left(\wp_{ai}(u), \frac{1}{2i}(e^{iv}\widetilde{\sigma}_{ai,i\xi}(u) - e^{-iv}\widetilde{\sigma}_{ai,-i\xi}(u)) \right).$$

We recall that, in the proof of Theorem 5.10.(II).(iv), we showed that $\beta_1(u, v) = (u, iv)$ is a locally $\mathbb{C}$-Nash isomorphism from $(\mathbb{C}^2, +, g_{6,i\xi,ai})$ to $(\mathbb{C}^2, +, g_{5,i\xi,ai})$. Hence, we get that $\mathrm{Aut}(\mathbb{C}^2, +, g_{6,i\xi,ai})$ equals to

$$\beta_1^{-1} \circ \mathrm{Aut}(\mathbb{C}^2, +, g_{5,i\xi,ai}) \circ \beta_1 = \{q(u, \xi \mathfrak{qc}(\Omega, q\Omega)u + v) \mid q \in \mathbb{Q}^*\},$$

which in turn must equal $\mathrm{Aut}(\mathbb{R}^2, +, g_{6,i\xi,ai})$, as required. □

Conclusions

Here we list some directions for further work:

1. Abelian surfaces: Our classification of two-dimensional abelian locally $\mathbb{K}$-Nash groups is provided up to isomorphism except in the case of abelian surfaces. The study of this case requires a more profound understanding of algebraic geometric techniques which are beyond the scope of this thesis. We cite C. Birkenhake and H. Lange [**4**] for complex abelian surfaces and B. H. Gross and J. Harris [**14**] and J. Huisman's [**22**] for real abelian surfaces as good sources for further research.

2. Study of the quotients: A more explicit description of the quotients of the simply connected two-dimensional abelian locally $\mathbb{K}$-Nash groups given in Proposition 3.10 could be obtained studying each possible discrete subgroup of $\mathbb{K}^2$.

3. Non-abelian version of Theorem 3.12: In this theorem we have showed that simply connected n-dimensional abelian locally $\mathbb{C}$-Nash groups are exactly the universal coverings of (abstract) abelian complex algebraic group of dimension n. We believe that the hypothesis of abelianity can be removed following E. Hrushovski and A. Pillay's proof of the analogous statement for locally Nash groups (see [**19**]) via a modification of Hrushovski's group configuration theorem (see [**18**]).

4. Classification of two-dimensional abelian Nash groups: This could be done in two steps. The first one should be to determine which locally Nash groups are Nash groups. The second one, determining which locally Nash isomorphisms between Nash groups are Nash isomorphisms.

5. Classification of two-dimensional abelian real algebraic groups: This could be done in two steps. The first one should be to determine which locally Nash groups are affine. These locally Nash groups are finite coverings of the connected component of real algebraic groups. The second one, determining the isomorphisms between real algebraic groups.

6. Definability of the charts of the classifications: In [**34**], Y. Peterzil and S. Starchenko proved that, when restricted to an adequate domain, the Weierstrass $\wp$-functions are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. This result was essential in J. Pila's proof of the André-Oort conjecture [**36**]. Later, in [**35**], Peterzil and Starchenko obtained similar results for some classical maps from the theory of abelian varieties and their moduli spaces. We expect that similar results can be proved for the functions that provide the charts of our classifications of locally $\mathbb{K}$-Nash groups. It will be also interesting to study if the same results could be obtained for any map admitting an AAT. See also [**23**].

7. Extension theorem for real closed field: It will be interesting to check if Theorem 1.11.(1) can be generalized to arbitrary real closed fields. To do this, the notion of analytic function must be substituted by the notion of K-holomorphic function in the sense of Y. Peterzil and S. Starchenko's work in [**32, 33**].

Conclusiones

A continuación listamos algunas direcciones para trabajo futuro:

1. Superficies abelianas: Nuestra clasificación de grupos localmente $\mathbb{K}$-Nash abelianos y de dimensión 2 está dada módulo isomorfismo salvo en el caso de superficies abelianas. El estudio de este caso requiere un conocimiento más profundo de técnicas de geometría algebraica que están más allá de los objetivos de esta tesis. Citamos a C. Birkenhake and H. Lange [**4**] para superficies abelianas complejas, y a B. H. Gross y J. Harris [**14**] y J. Huisman's [**22**] para superficies abelianas reales, como buenas referencias en las que basar futuras investigaciones.

2. Estudio de los cocientes: Una descripción más explícita de los cocientes de los grupos localmente $\mathbb{K}$-Nash abelianos, simplemente conexos y de dimensión 2 puede obtenerse estudiando cada posible subgrupo discreto de $\mathbb{K}^2$.

3. Versión no abeliana del Teorema 3.12: En este teorema hemos probado que los grupos localmente $\mathbb{K}$-Nash abelianos, simplemente conexos y de dimensión n son exactamente los recubridores universales de los grupos algebraicos complejos (abstractos) de dimensión n. Creemos que la hipótesis de abelianidad puede eliminarse, adaptando la demostración de E. Hrushovski y A. Pillay del resultado análogo para grupos localmente Nash (véase [**19**]) vía una modificación del teorema de configuración de grupo de Hrushovski (véase [**18**]).

4. Clasificación de los grupos de Nash abelianos de dimensión 2*:* Se podría hacer en dos pasos: El primero debería ser determinar qué grupos localmente Nash son grupos de Nash. El segundo, determinar cuáles de los isomorfismos localmente Nash entre grupos de Nash son isomorfismos de Nash.

5. Clasificación de los grupos algebraicos reales abelianos de dimensión 2*:* Se podría hacer en dos pasos: El primero debería ser determinar cuáles de los grupos localmente Nash son afines. Los grupos localmente Nash afines son los recubridores finitos de las componentes conexas (de la identidad) de grupos reales algebraicos. El segundo paso debería ser determinar los isomorfismos entre los grupos reales algebraicos.

6. Definibilidad de las cartas de la clasificación: En [**34**], Y. Peterzil y S. Starchenko demuestran que, al restringirse a un dominio adecuado, las funciones $\wp$ de Weierstrass son definibles en la estructura o-minimal $\mathbb{R}_{\mathrm{an,exp}}$.

Este resultado es esencial en la demostración de J. Pila de la conjetura de André-Oort [**36**]. Más tarde, en [**35**], Peterzil y Starchenko obtienen resultados similares para algunas funciones clásicas de la teoría de variedades abelianas y sus espacios de moduli. Creemos que resultados similares a los de Peterzil y Starchenko pueden ser obtenidos para las cartas de nuestras clasificaciones de grupos localmente $\mathbb{K}$-Nash. También sería interesante estudiar si el mismo tipo de resultados puede obtenerse para cualquier función que satisfaga un TAA. Véase también [**23**].

7. Teorema de extensión para cuerpos realmente cerrados: Sería interesante comprobar si la demostración del Teorema 1.11.(1) puede ser generalizada para cuerpos realmente cerrados arbitrarios. Para ello, la noción de función analítica debe ser sustituida por la noción de función K-holomorfa, en el sentido utilizado por Y. Peterzil y S. Starchenko's en [**32, 33**].

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